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Continuous Orthonormal Bases and Spectral Synthesis

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Physicists describe the Fourier transform as a "change of bases" between the Dirac basis and the Fourier basis. Due to Plancherel's Theorem it defines a unitary automorphism of the Hilbert space $L^2(\mathbb{R}^d)$. This situation seems to suggest to call both of these systems "continuous orthonormal bases" (despite the lack of a good, formal definition of such a concept). However, taking a closer look both systems are far from a basis because they are neither a minimal family nor do their constituting elements belong to the Hilbert space.

Taking the view-point of the "Banach Gelfand Triple" (S_0, L^2, S_0^*), consisting of the Feichtinger Algebra $S_0(\mathbb{R}^d)$, the Hilbert space $L^2(\mathbb{R}^d)$ and the dual space of *mild distributions* $S_0^*(\mathbb{R}^d)$, one can describe both systems as the limits of Gabor systems which constitute redundant resolutions of the identity, so-called tight Gabor frames. Starting from this description we can also describe other "mild bases" which allow to describe among others also the Fractional Fourier Transform as such a change of basis (corresponding to rotation in phase space).

Finally we will link the concepts of *spectral analysis* and *spectral synthesis* for mild distributions to the classical approach, making use of the $L^1 - L^\infty$-duality. In short, a set E in the spectral domain is a set of spectral synthesis if any mild distributions with $\mathrm{spec}(\sigma) = \mathrm{supp}(\widehat{\sigma}) \subseteq E$ can be approximated by trigonometric polynomials with frequencies from E (only), in the sense of mild distributions.