Overview of Fuglede's conjecture on cyclic groups

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Tiling versus Spectrality

Let G be a locally compact abelian group and $\widehat{G}$ denotes its dual. Let μ denote the Haar measure on G and let $S \subseteq G$ be a subset with $0 < \mu(S) < \infty$.

- A set $\Lambda \subset \widehat{G}$ is called the **spectrum** of S, if the characters $\{\lambda\}_{\lambda \in \Lambda}$ gives an orthogonal basis of $L^2(S)$. If there exists such a set S, then S is called a **spectral** set.
- S is called a **tile** of G, if there exists a set $T \subseteq G$ such that up to a set of measure zero every element of G can be uniquely written of the form $s + t$, where $s \in S, t \in T$. This we denote by $S + T = G$.

Fuglede's conjecture and his results

Conjecture (Fuglede '74)

Let $S \subset \mathbb{R}^d$ be a bounded measurable set with $\lambda(S) > 0$. Then S tiles $\mathbb{R}^d$ by translation if and only if S is spectral. The spectral sets and tiles coincide in $\mathbb{R}^d$.

Theorem (Fuglede '74)

- *If $S \subset \mathbb{R}^d$ is a tile whose tiling partner is a lattice, then S is spectral.*
- *If $S \subset \mathbb{R}^d$ is spectral with spectrum which is a lattice, then S tiles $\mathbb{R}^d$.*
- *Disc and triangle in $\mathbb{R}^2$ is neither spectral, nor a tile.*

Conjecture (Fuglede's conjecture on finite abelian groups)

Let G be a finite abelian group and $S \subset G$. Then S is a spectral if and only if S is a tile.

Fuglede's conjecture in $\mathbb{R}^d$: Spectral $\Rightarrow$ Tile direction

The conjecture has been disproved.

Theorem

- *Tao '04: There is a spectral set of size* 6 *in* $\mathbb{Z}_3^5$*, hence Fuglede's conjecture is not true in* $\mathbb{R}^d$*, if* $d \geqslant 5$*.*
- *Matolcsi '05: There is a spectral set of size* 6 *in* $\mathbb{Z}_3^4$*, hence Fuglede's conjecture is not even true in* $\mathbb{R}^4$*.*
- *Kolountzakis-Matolcsi '06: There is a spectral set of size* 6 *in* $\mathbb{Z}_8^3$*, hence Fuglede's conjecture is not even true in* $\mathbb{R}^3$*.*

For convex domains positive result is true by Lev and Matolcsi '22.

Theorem

Fuglede's conjecture holds on convex domains of $\mathbb{R}^d$*.*

Tile $\Rightarrow$ Spectral direction

Lagarias and Wang '97 posed the following conjecture.

Conjecture (Universal Spectral Set Conjecture (USSC))

If a set S tiles a group with tiling partners $T_1, \ldots, T_n$, then they are spectral and they have a common spectrum.

Theorem

- *Kolounztakis-Matolcsi '05: In $\mathbb{Z}_6^5$ USSC fails. This implies that Tile$\Rightarrow$Spectral direction fails (**T**$\nrightarrow$ **S**) in $\mathbb{R}^d$ ($d \geqslant 5$).*
- *Farkas-Révész '06: In $\mathbb{Z}_6^4$ USSC fails. This implies **T**$\nrightarrow$ **S** in $\mathbb{R}^4$.*
- *Farkas-Matolcsi-Móra '06: In $\mathbb{Z}_{24}^3$ USSC fails. This implies **T**$\nrightarrow$ **S** in $\mathbb{R}^3$.*

Open questions

Open Question

Does Fuglede's conjecture hold in $\mathbb{R}$ *and* $\mathbb{R}^2$*?*

The question on $\mathbb{R}$ is closely connected to analogue question on cyclic groups; many subcases are known.
The question on $\mathbb{R}^2$ is widely open; a few subcases are known.

Open Question

What are those finite abelian groups where Fuglede's conjecture holds?

Fuglede's conjecture on finite Abelian groups

Conjecture (Fuglede's conjecture on finite Abelian groups)

Let S tiles G by translation if and only if the functions on S has an orthogonal basis of characters of G.

Every counterexample mentioned is based on the following. If we have a counterexample on a finite abelian group with d generators, then it can be extended to a counterexample in $\mathbb{Z}^d$ and blow it up to one in $\mathbb{R}^d$.
Denoting $\mathbf{S}-\mathbf{T}(G)$ (resp. $\mathbf{T}-\mathbf{S}(G)$), if the *Spectral* $\Rightarrow$ *Tile* (resp. *Tile* $\Rightarrow$ *Spectral*) direction of Fuglede's conjecture holds in G. We have the following implication:

$$\mathbf{T}-\mathbf{S}(\mathbb{R}^d) \Longrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}^d) \Longrightarrow \mathbf{T}-\mathbf{S}(G_d),$$

$$\mathbf{S}-\mathbf{T}(\mathbb{R}^d) \Longrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}^d) \Longrightarrow \mathbf{S}-\mathbf{T}(G_d),$$

where G_d can be any Abelian group of d generators.

One dimensional case

For $\mathbf{T}-\mathbf{S}$ direction we can say more in the one dimensional case. The following was presented by Dutkay and Lai.

$$\mathbf{T}-\mathbf{S}(\mathbb{R}) \Longleftrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}) \Longleftrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}_{\mathbb{N}}).$$

Conjecture

$$\mathbf{S}-\mathbf{T}(\mathbb{R}) \Longleftrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}) \Longleftrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}_{\mathbb{N}}).$$

Cyclic group case I.

I just summarized the top results in each direction, but many of them has a longer history.

Theorem

Fuglede's conjecture holds in the following groups:

- $\mathbb{Z}_{p^n q^m}$, *where* p *and* q *are distinct primes.* $n \in \mathbb{N}, 0 \leqslant m \leqslant 6 \in \mathbb{N}$ *(R. Malikiosis, 2022),*
- $\mathbb{Z}_{pqrs}$, *where* p, q, r *and* s *are distinct primes. (GK, R.Malikiosis, G. Somlai, M. Vizer, 2022),*
- $\mathbb{Z}_{p^n qr}$, *where* p, q *and* r *are distinct primes,* $n \in \mathbb{N}$. *(T. Zhang, 2023).*

Cyclic group case II.

Theorem

Every tile is spectral in

- $\mathbb{Z}_{p^n q^m}$, *where* p *are* q *are distinct primes (E. Coven, A. Meyerovitz, 1999),*
- $\mathbb{Z}_{p_1^n p_2 \dots p_k}$, *where* p_i *are distinct primes for all* $1 \leqslant i \leqslant k$ *(R. Malikiosis, 2022),*
- $\mathbb{Z}_{p^2 q^2 r^2}$, *where* p, q *and* r *are distinct primes (I. Londner, I. Laba, 2023).*

Theorem

Fuglede's conjecture holds in $\mathbb{Z}_{p^2 q^2 r}$ *if* $r > p^2 q^2$ *(T. Fallon, GK, A. Mayeli, G. Somlai, 2023+).*

Mask polynomial

The *mask polynomial* of S on a cyclic group $\mathbb{Z}_N$ is defined as $m_S(x) = \sum_{s \in S} x^s$ in $\mathbb{Z}[x]/(x^N - 1)$.

Proposition

If $k \mid N$ and a character of order k vanishes on $S \subseteq \mathbb{Z}_N$, i.e., $\sum_{x \in S} \chi(x) = 0$, then every character of order k vanishes on S, i.e. $\Phi_k \mid m_S$.

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Our aim is to understand which divisibility relations hold for spectral sets:
Every element $0 \neq x \in S - S$ of a spectral set S implies a divisibility condition.
We notice that $\Phi_{p^{k_1}} \ldots \Phi_{p^{k_i}} \mid m_S$ implies $p^i \mid |S|$ for every prime p and $k_i \in \mathbb{N}, i \in \mathbb{N}$.

Coven-Meyerowitz properties

Let H_S be the set of prime powers p^a dividing N such that $\Phi_{p^a}(x) \mid m_S(x)$.

(T1) $m_S(1) = |S| = \prod_{d \in H_S} \Phi_d(1)$.

(T2) For pairwise relative prime elements s_i of H_S, $\Phi_{\prod s_i} \mid m_S$.

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Theorem

- *(Coven-Meyerowitz) If the properties (T1) and (T2) hold for m_S, then S tiles $\mathbb{Z}$.*
- *If the size of a tile is $p^n q^m$, then (T1) and (T2) hold.*
- *(Łaba) If (T1) and (T2) hold for m_S, then S is a spectral set.*

Corollary

Every tile is spectral in $\mathbb{Z}_{p^n q^m}$ $(n, m \in \mathbb{N})$.

Reduction of the problem

Lemma

Suppose that S be a spectral set. Then the followings hold true.

- *WLOG we can assume that $0 \in S$ and $0 \in \Lambda$.*
- *If S is contained in a subgroup H of G, where $\mathbf{S} - \mathbf{T}(H)$ holds, then S is a tile.*
- *If Λ is contained in a subgroup H of G, where $\mathbf{S} - \mathbf{T}(H)$ holds, then S is a tile.*

Lemma

If S is the union of $\mathbb{Z}_p$ cosets, then S tiles G.

Condition on r

The following fact is from Ruxi Shi (Fourier bases 2018 meeting, Crete)

Lemma

Let S be a spectral set in $\mathbb{Z}_N$. Assume that for every divisor $d \mid N$ we have $\mathbf{S} - \mathbf{T}(\mathbb{Z}_d)$. Suppose that for a prime $r \mid N$ and $r \nmid |S|$ the following implication holds

$$\Phi_{mr} \mid m_S \Longrightarrow \Phi_m \mid m_S,$$

whenever $m \mid N$ is coprime to p. Then S is a tile.

Otherwise we get the following.

Lemma

Let $r \mid N$ be a prime and $m \mid N$ is such that $\gcd(m, r) = 1$ and $\Phi_{mr} \mid m_S$ but $\Phi_m \nmid m_S$. Then each $\mathbb{Z}_{\frac{N}{r}}$ coset contains at least one point of S. Thus $|S| \geqslant r$.

If further $r > p^2q^2$, then $\Phi_r \mid m_\Lambda$. Hence $r \mid |S|$.

Sketch of the proof

Theorem

Every spectral set of $\mathbb{Z}_{p^2q^2r}$ tiles if $r \geqslant p^2q^2$.

We distinguish cases according the divisors of p^2q^2r. We say that

$$d \,||\, |S|, \text{ if } \gcd(p^2q^2r, |S|) = d.$$

By this reasons we distinguish three cases.

- Small sets: $d \,||\, |S|$, where $r \nmid d$: These cases are excluded.
- Middle size sets: $r \,||\, |S|$, $pr \,||\, |S|$, $qr \,||\, |S|$, or $pqr \,||\, |S|$.
- Large sets: $p^2r \,||\, |S|$, $q^2r \,||\, |S|$, $p^2qr \,||\, |S|$, $pq^2r \,||\, |S|$, or $p^2q^2r \,||\, |S|$ (also $p^2q^2 \,||\, |S|$): Pigeonhole principle for the cardinality of the set implies strict role how S should look like.

Middle sets I.: mod p method

Lemma

If $n = p^k m$, where p is a prime and $p \nmid m$, then

$$\Phi_n \mid m_S \text{ in } \mathbb{Z}[x] \Longrightarrow \Phi_m \mid m_S \text{ in } \mathbb{Z}_p[x].$$

Lemma

Let $n = p^k m$, where p is a prime and $p \nmid m$, and S is a subset of $\mathbb{Z}_n$. Then the following implication holds. If

$$\Phi_d \mid m_S \text{ in } \mathbb{Z}_p[x] \quad \forall d \mid m, \tag{1}$$

then $|S| = km + lp$ for nonnegative $k, l \in \mathbb{Z}$.

Middle sets II.: Main strategy

We have $\Phi_r \mid m_\Lambda$. We assume $pqr \,\|\, |S|$ (the most problematic case).

$$\begin{array}{ccccccc}
\Phi_p & \text{or} & \Phi_{pq} & \text{or} & \Phi_{pq^2} \mid m_\Lambda & \Longrightarrow & \Phi_p \mid m_\Lambda \text{ in } \mathbb{Z}_q[x] \\
\Phi_{pr} & \text{or} & \Phi_{pqr} & \text{or} & \Phi_{pq^2r} \mid m_\Lambda & \Longrightarrow & \Phi_{pr} \mid m_\Lambda \text{ in } \mathbb{Z}_q[x] \\
\Phi_{p^2} & \text{or} & \Phi_{p^2q} & \text{or} & \Phi_{p^2q^2} \mid m_\Lambda & \Longrightarrow & \Phi_{p^2} \mid m_\Lambda \text{ in } \mathbb{Z}_q[x] \\
\Phi_{p^2r} & \text{or} & \Phi_{p^2qr} & \text{or} & \Phi_{p^2q^2r} \mid m_\Lambda & \Longrightarrow & \Phi_{p^2r} \mid m_\Lambda \text{ in } \mathbb{Z}_q[x]
\end{array}$$

Table: System of divisibility relations

Now we apply Lemma: $|\Lambda| = |S| = kp^2r + lq$, for some $0 \leqslant k, l \in \mathbb{N}$.

- If $k \geqslant 1$ and $l \geqslant 1$, then S is large.
- If $l = 0$, the S is a tile or large.
- If $k = 0$, then $S \cap (x + \mathbb{Z}_{q^2})$ can be: $\emptyset$, a $\mathbb{Z}_q$ coset, or a full coset representative system of $\mathbb{Z}_q$ in $\mathbb{Z}_{q^2}$. By cardinality, each nonempty intersection is of one type. Similarly idea for p implies strict rules for S, and hence S tiles.

Middle sets III.: Geometric argument for the excluded cases

If this system of divisibility relations does not hold, then at least a row of conditions fails simultaneously. For instance,

$$\Phi_p \nmid m_\Lambda \quad \text{and} \quad \Phi_{pq} \nmid m_\Lambda \quad \text{and} \quad \Phi_{pq^2} \nmid m_\Lambda .$$

In each $\mathbb{Z}_{pq^2}$ coset at most one $\mathbb{Z}_{q^2}$ coset contain elements of S.

$$|S \cap (x + \mathbb{Z}_{q^2})| \leqslant q \Longrightarrow |S \cap (x + \mathbb{Z}_{pq^2})| \leqslant q \Longrightarrow |S \cap \mathbb{Z}_{p^2q^2r}| \leqslant pqr .$$

Since $pqr \mid |S|$ we have $|S| = pqr$ and all $\leqslant$ is $=$.
We obtain that each $\mathbb{Z}_{pq^2}$ coset contains exactly one $\mathbb{Z}_{q^2}$ coset having q elements of S of a given type. Hence S is a tile.

Thank you for your attention!

End