Topologically transitive operators on the space of Radon measures

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Stefan Ivković

Mathematical Institute of the Serbian Academy of Sciences and Arts, p.p. 367, Kneza Mihaila 36, 11000 Beograd, Serbia

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If $\mathcal{X}$ is a Banach space, the set of all bounded linear operators from $\mathcal{X}$ into $\mathcal{X}$ is denoted by $B(\mathcal{X})$. Also, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition

Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *topologically transitive* if for each non-empty open subsets U, V of $\mathcal{X}$, $T_n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$.
A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *topologically hyper-transitive* if for each non-empty open subsets U, V of $\mathcal{X}$, there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that $T_{n_k}(U) \cap V \neq \varnothing$ for all $k \in \mathbb{N}$.
A single operator T in $B(\mathcal{X})$ is called topologically transitive (respectively hyper-transitive) if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is topologically transitive (respectively hyper-transitive).

Definition

[**tsi**] Let $\mathcal{X}$ be a Banach space, and $(T_n)_{n\in\mathbb{N}_0}$ be a sequence of operators in $B(\mathcal{X})$. A vector $x \in \mathcal{X}$ is called a *periodic element* of $(T_n)_{n\in\mathbb{N}_0}$ if there exists a constant $N \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $T_{kN}x = x$. The set of all periodic elements of $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$. The sequence $(T_n)_{n\in\mathbb{N}_0}$ is called *chaotic* if $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive and $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$ is dense in $\mathcal{X}$. An operator $T \in B(\mathcal{X})$ is called *chaotic* if the sequence $\{T^n\}_{n\in\mathbb{N}_0}$ is chaotic.

Definition

[**tsi**] Let X be a topological space. Let $\alpha : X \longrightarrow X$ be invertible, and α, α^{-1} be Borel measurable. We say that α is *aperiodic* if for each compact subset K of X, there exists a constant $N > 0$ such that for each $n \geq N$, we have $K \cap \alpha^n(K) = \varnothing$, where α^n means the n-fold combination of α.

We let Ω be a locally compact non-compact Hausdorff space and α be an aperiodic homeomorphism of Ω. As usual, $C_0(\Omega)$ denotes the space of all continuous functions on Ω vanishing at infinity, $C_b(\Omega)$ denotes the space of bounded, continuous functions on Ω, whereas $C_c(\Omega)$ stands for the set of all continuous, compactly supported functions on Ω. Both $C_0(\Omega)$ and $C_b(\Omega)$ are equipped with the supremum norm. Moreover, we let w be a positive continuous bounded function on Ω such that also $w^{-1} \in C_b(\Omega)$ and we put then $T_{\alpha,w}$ to be the weighted composition operator on $C_0(\Omega)$ with respect to α and w, that is $T_{\alpha,w}(f) = w \cdot (f \circ \alpha)$ for all $f \in C_0(\Omega)$. Easily, one can see that by the above assumptions $T_{\alpha,w}$ is well-defined and $\|T_{\alpha,w}\| \leq \|w\|_{\sup}$. Since $\frac{1}{w}$ is also bounded, then $T_{\alpha,w}$ is invertible and we have

$$T_{\alpha,w}^{-1} f = \frac{f \circ \alpha^{-1}}{w \circ \alpha^{-1}}, \quad (f \in C_0(\Omega)).$$

Simply we denote $S_{\alpha,w} := T_{\alpha,w}^{-1}$.

Remark

If w and $\frac{1}{w}$ are weights, the inverse of a weighted composition operator $T_{\alpha,w}$ is also a weighted composition operator. In fact, $S_{\alpha,w} = T_{\alpha^{-1},\frac{1}{w\circ\alpha^{-1}}}$. Moreover, if T_{α_1,w_1} and T_{α_2,w_2} are two weighted composition operators, then

$$T_{\alpha_2,w_2} \circ T_{\alpha_1,w_1} = T_{\alpha_1\circ\alpha_2,w_2(w_1\circ\alpha_2)},$$

so the composition of two weighted composition operators is again a weighted composition operator. By some calculation one can see that for each $n \in \mathbb{N}$ and $f \in C_0(\Omega)$,

$$T^n_{\alpha,w} f = \left(\prod_{j=0}^{n-1} (w \circ \alpha^j)\right) \cdot (f \circ \alpha^n) \ (1)$$

and

$$S^n_{\alpha,w} f = \left(\prod_{j=1}^{n} (w \circ \alpha^{-j})\right)^{-1} \cdot (f \circ \alpha^{-n}). \ (2)$$

Lemma

The following are equivalent.

(i) $T_{\alpha,w}$ is topologically hyper-transitive on $C_0(\Omega)$.

(ii) For every compact subset K of $\mathbb{R}$ there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty}(\sup_{t\in K}|\prod_{j=0}^{n_k-1}(w\circ\alpha^{j-n_k})(t)|) = \lim_{k\to\infty}(\sup_{t\in K}|\prod_{j=0}^{n_k-1}(w\circ\alpha^{j})^{-1}(t)|) = 0.$$

The adjoint $T^*_{\alpha,w}$ is a bounded operator on $M(\Omega)$ where $M(\Omega)$ stands for the Banach space of all regular Borel measures on Ω equipped with the total variation norm. It is straightforward to check that

$$T^*_{\alpha,w}(\mu)(E) = \int_E w \circ \alpha^{-1} \, d\mu \ \circ \alpha^{-1}$$

for every $\mu \in M(\Omega)$, and every measurable subset E od Ω. By (1) and (2) it follows then that for every $n \in \mathbb{N}$, $\mu \in M(\Omega)$ and a Borel measurable subset $E \subseteq \Omega$ we have that

$$T^{*n}_{\alpha,w}(\mu)(E) = \int_E \prod_{j=0}^{n-1} w \circ \alpha^{j-n} \, d\mu \ \circ \alpha^{-n}$$

and

$$S^{*n}_{\alpha,w}(\mu)(E) = \int_E \prod_{j=1}^{n} (w \circ \alpha^{n-j})^{-1} \, d\mu \ \circ \alpha^{n}.$$

Proposition

The following statements are equivalent.

i) $T^*_{\alpha,w}$ is topologically hyper-transitive on $M(\Omega)$.

ii) For every compact subset K of Ω and any two measures μ, v in $M(\Omega)$ with $|v|(K^c) = |\mu|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{A_k\}$, $\{B_k\}$ of Borel subsets of K such that $\alpha^{n_k}(K) \cap K = \varnothing$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in K\cap A_k^c} (\prod_{j=0}^{n_k-1} (w \circ \alpha^j)(t)) = \lim_{k\to\infty} \sup_{t\in K\cap B_k^c} (\prod_{j=1}^{n_k} (w \circ \alpha^{-j})^{-1}(t)) = 0.$$

Corollary

We have that ii) $\Rightarrow$ i)
i) $T^*_{\alpha,w}$ is topologically hyper-transitive on $M(\Omega)$.
ii) For every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \sup_{t\in K}\Big(\prod_{j=0}^{n_k-1}(w\circ\alpha^j)(t)\Big) = \lim_{k\to\infty} \sup_{t\in K}\Big(\prod_{j=1}^{n_k}(w\circ\alpha^{-j})^{-1}(t)\Big) = 0.$$

Open problem
Does there exist an example where the equivalent conditions of the part ii) in the previous proposition are satisfied, whereas the sufficient conditions of the part ii) in this corollary are not satisfied ?

For each $n \in \mathbb{N}$, we set now $C^{*(n)}_{\alpha,W} = \frac{1}{2}(T^{*n}_{\alpha,W} + S^{*n}_{\alpha,W})$.

Proposition

We have that $(ii) \Rightarrow (i)$:

(i) The sequence $(C^{*^{(n)}}_{\alpha,w})$ is topologically hyper-transitive on $M(\Omega)$.

(ii) For every compact subset K od Ω and any two measures μ, v in $M(\Omega)$ with $|\mu|(K^c) = |v|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ and sequences $\{A_k\}_k, \{F_k\}_k, \{D_k\}_k$ of Borel subsets of K such that

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{n\to\infty} |v|(A_k) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in K\cap A_k^c} \left(\prod_{j=0}^{n_k-1}(w\circ\alpha^j)(t)\right) = \lim_{k\to\infty} \sup_{t\in K\cap A_k^c} \left(\prod_{j=0}^{n_k-1}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in F_k} \left(\prod_{j=0}^{2n_k-1}(w\circ\alpha^j)(t)\right) = \lim_{k\to\infty} \sup_{t\in D_k} \left(\prod_{j=1}^{2n_k}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

where $F_k \cap D_k = \varnothing$ and $A_k^c \cap K = F_k \cup D_k$ for all k.

Corollary

We have that $(ii) \Rightarrow (i)$:

(i) *The sequence* $(C_{\alpha,w}^{*(n)})$ *is topologically hyper-transitive on* $M(\Omega)$.

(ii) *For every compact subset* K *od* Ω *we have that*

$$\lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{n-1}(w\circ\alpha^j)(t)\right) = \lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{n-1}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

$$\lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{2n-1}(w\circ\alpha^j)(t)\right) = \lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=1}^{2n}(w\circ\alpha^{-j})^{-1}(t)\right) = 0.$$

Proposition

We have that $ii) \Rightarrow i)$.

(i) The sequence $(C^{*^{(n)}}_{\alpha,w})_n$ is chaotic on $M(\Omega)$.

(ii) For any compact subset K of Ω and any measure $\mu \in M(\Omega)$ with $|\mu|(K^c) = 0$, there exist a sequence of Borel subsets $\{A_k\}_k$ of K and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that $\lim_{k\to\infty} |\mu|(A_k^c) = 0$ and

$$\lim_{k\to\infty} \sum_{l=1}^{\infty} \sup_{t\in K\cap A_k^c} \Big(\prod_{j=0}^{ln_k-1} (w\circ\alpha^j)(t)\Big) = \lim_{k\to\infty} \sum_{l=1}^{\infty} \sup_{t\in K\cap A_k^c} \Big(\prod_{j=1}^{ln_k} (w\circ\alpha^{-j})^{-1}(t)\Big) = 0,$$

where the corresponding series are convergent for each k.

Corollary

We have that ii) $\Rightarrow$ i).
(i) The sequence $(C^{*^{(n)}}_{\alpha,w})_n$ is chaotic on $M(\Omega)$.
(ii) For any compact subset K of Ω we have that

$$\lim_{n\to\infty}\sum_{l=1}^{\infty}\sup_{t\in K}\ (\prod_{j=0}^{ln-1}(w\circ\alpha^j)(t)) = \lim_{n\to\infty}\sum_{l=1}^{\infty}\sup_{t\in K}\ (\prod_{j=1}^{ln}(w\circ\alpha^{-j})^{-1}(t)) = 0,$$

where the corresponding series are convergent for each n.

Example

Let $\Omega = \mathbb{R}, \ \alpha(t) = t + 1$ for all $t \in \mathbb{R}$ and

$$w(t) = \begin{cases} 2 & \text{for } t \leq -1, \\ \frac{1}{2} & \text{for } t \geq 1, \\ \text{linear on the segment } [-1, 1]. \end{cases}$$

In this case, the sufficient conditions of the preceding corollaries are satisfied.

In general, if $M, \epsilon > 0$ such that $1 + \epsilon < M$, $1 - \epsilon > \frac{1}{M}$, and $K_1, K_2 > 0$, then if $w \in C_b(\mathbb{R})$ satisfies that $M \geq |w(t)| \geq 1 + \epsilon$ for all $t \leq -K_1$ and $\frac{1}{M} \leq |w(t)| \leq 1 - \epsilon$ for all $t \geq K_2$, then the sufficient conditions of the preceding corollaries are satisfied.

We recall that a subset S of a Banach space Y is called *spaceable* in Y if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of Y.
In this presentation, a subset B of a vector space Y is called a cone if for each scalar c, $cB \subseteq B$.
For a Borel measurable subset E and some $\mu \in M(\Omega)$, we let μ_E be the measure given by $\mu_E(B) := \mu(B \cap E)$ for every Borel subset B of Ω. If K is a cone in $M(\Omega)$, we denote

$$\widetilde{K} := \{\mu_E : \mu \in K, \, E \text{ is Borel}\}.$$

Since for every scalar $\lambda \in \mathbb{C}$ we have $(\lambda\mu)_E = \lambda \, \mu_E$, it follows that $\widetilde{K}$ is a cone. Moreover, $K \subseteq \widetilde{K}$.

Proposition

Let K be a cone in $M(\Omega)$. If there exists a sequence of mutually disjoint Borel subsets $\{E_n\}_{n\in\mathbb{N}}$ of Ω such that for all n

$$\{\mu_{E_n} : \mu \in \widetilde{K}\} \neq \{\mu_{E_n} : \mu \in M(\Omega)\},$$

then $M(\Omega) \setminus \tilde{K}$ (and consequently $M(\Omega) \setminus K$) is spaceable in $M(\Omega)$.

Corollary

Let K be the cone of all scalar multiples of positive Radon measures on a non-compact, locally compact Hausdorff space Ω. Then $M(\Omega) \setminus K$ is spaceable in $M(\Omega)$.

Thank you for attention !

stefan.iv10@outlook.com

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