Moment and derivations on hypergroups

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Agenda

Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X\colon \Omega \to \mathbb{R}$ be an integrable random variable. Then the quantity

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$$

is called the expectation of X. If $\mu_X\colon \mathcal{B} \to \mathbb{R}$ is the law of a random variable X, i.e.

$$\mu_X(B) := \mathbb{P}\left(\{\omega \in \Omega : X(\omega) \in B\}\right), \quad B \in \mathcal{B}(\mathbb{R})$$

then the expectation is given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} x^1 d\mu_X(x).$$

Variance

Assume now that $X\colon \Omega \to \mathbb{R}$ be a square integrable random variable. Then the quantity

$$\begin{aligned}\mathrm{Var}[X] &= \int_\Omega [X(\omega) - \mathbb{E}[X]]^2 \, d\mathbb{P}(\omega) \\ &= \int_\Omega X(\omega)^2 d\mathbb{P}(\omega) - \mathbb{E}[X]^2\end{aligned}$$

is called the variance of X. Then

$$\mathrm{Var}[X] = \int_{\mathbb{R}} x^2 d\mu_X(x) - \left[\int_{\mathbb{R}} x^1 d\mu_X(x)\right]^2,$$

where μ_X denotes the law of a random variable X.

Properties

Observe now that functions $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ given by

$$g_1(x) = x^1 \quad g_2(x) = x^2$$

satisfy

$$g_1(x + y) = g_1(x) + g_1(y)$$

and

$$g_2(x + y) = g_2(x) + 2g_1(x)g_1(y) + g_2(y).$$

Classical moment problem

For a given sequence $(s_n)_{n\in\mathbb{N}_0}$ of real numbers find the neccesarily and sufficient conditions for the existence of a measure μ on $[0,+\infty)$ such that

$$s_n = \int_0^\infty x^n \mu(x), \quad n \in \mathbb{N}_0.$$

Sequence of moment functions

Let now $g_i\colon \mathbb{R} \to \mathbb{R}$ for $i = 0, 1, \ldots, N$ be given by

$$g_0(x + y) = g_0(x)g_0(y)$$

and

$$\int_{\mathbb{R}} g_k(z)d(\delta_x * \delta_y) = g_k(x + y) = \sum_{i=0}^{k} \binom{k}{i} g_i(x)g_{k-i}(y),$$

where δ_x is the Dirac measure concentrated at the point x and

$$\delta_x * \delta_y := \delta_{x+y}.$$

Then

$$\mathbb{E}[X^k] = \mathbb{E}[g_k(X)] = \int_{\Omega} X^k(\omega)d\mathbb{P}(\omega)$$

is k - moment of a r.v. X.

Moment function (of rank one)

Let N be a nonnegative integer. A function $\varphi\colon G \to \mathbb{C}$ is called a *moment function of order N*, if there exist functions $\varphi_k\colon G \to \mathbb{C}$ such that $\varphi_0 = 1$, $\varphi_N = \varphi$ and

$$\varphi_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\varphi_{k-j}(y) \tag{1}$$

for x, y in G and $k = 0, 1, \dots, N$.
For higher rank: “changing indecies to multi-indecies” (much more compicated in practice)

Moment function vs functions of binomial type

In [1] it was shown that that if $(G,+)$ is a grupoid and R is a commutative ring, then functions $\varphi_n\colon G\to R$ satisfying (1) for each n in $\mathbb{N}$ are of the form

$$\varphi_n(t)=n!\sum_{j_1+2j_2+\cdots+nj_n=n}\prod_{k=1}^{n}\frac{1}{j_k!}\left(\frac{a_k(t)}{k!}\right)^{j_k} \tag{2}$$

for all t in G and k in $\mathbb{N}$ and arbitrary homomorphisms a_k from $(G,+)$ into $(R,+)$.

Bell polynomials

Let us consider the sequence of complex polynomials $(B_n)_{n\in\mathbb{N}}$ defined through the following recurrence: for each $t, t_1, t_2, \ldots, t_{n+1}$ in $\mathbb{C}$ we let

$$B_0(t) = 1,$$
$$B_{n+1}(t_1, \ldots, t_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(t_1, \ldots, t_{n-i}) t_{i+1}$$

for each n in $\mathbb{N}$. Alternatively, we can also use the double series expansion of the generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{t^n}{n!}.$$

We call B_n the n^{th} *complete (exponential) Bell polynomial*.

Theorem (Ż.F., E. Gselmann and L. Székelyhidi [4])

Let G be a commutative group, r a positive integer, and for each α in $\mathbb{N}^r$, let $f_\alpha\colon G \to \mathbb{C}$ be a function. If the sequence of functions $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ forms a generalized moment sequence of rank r, then there exists an exponential $m\colon G \to \mathbb{C}$ and a sequence of complex-valued additive functions $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ such that for every multi-index α in $\mathbb{N}^r$ and x in G we have

$$f_\alpha(x) = B_\alpha(a(x))m(x). \tag{3}$$

Measure algebra as $\mathcal{M}_c(X) = (\mathcal{C}(X))^*$

Let X be a commutative (hyper)group, let $\mathcal{C}(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}$. The dual $\mathcal{C}^*(X)$ can be identified with the *measure algebra* $\mathcal{M}_c(X)$ of X. The space $\mathcal{M}_c(X)$ of all compactly supported complex Borel measures on X with the addition and multiplication by complex numbers. The convolution of measures can be defined by

$$\int_X f\, d(\mu * \nu) = \int_X \int_X f(x * y)\, d\mu(x)\, d\nu(y), \quad f \in \mathcal{C}(X).$$

The measure algebra as a module over the ring $\mathcal{C}(X)$: the action of φ in $\mathcal{C}(X)$ on $\mathcal{M}_c(X)$ is defined by the multiplication of the measure μ by φ defined as

$$\langle \varphi \cdot \mu, f \rangle = \int_X f \varphi\, d\mu.$$

Derivation

A *derivation* of the measure algebra usually defined as a continuous linear operator $D : \mathcal{M}_c(X) \to \mathcal{M}_c(X)$ satisfying the additional property

$$D(\mu * \nu) = D\mu * \nu + \mu * D\nu \tag{4}$$

for each μ, ν in $\mathcal{M}_c(X)$. A modification: instead of linearity we require that D is a continuous module homomorphism of $\mathcal{M}_c(X)$ as a module over the ring of continuous functions $\mathcal{C}(X)$. In other words, besides (4) we assume that

$$D(\mu + \nu) = D\mu + D\nu \tag{5}$$

$$D(\varphi\mu) = \varphi D\mu \tag{6}$$

holds for each μ, ν in $\mathcal{M}_c(X)$ and φ in $\mathcal{C}(X)$. Thus (5) and (6) means that D is a module homomorphism.

Higher order derivations

Suppose that r is a positive integer. The family $(D_\alpha)_{\alpha\in\mathbb{N}^r}$ of continuous module homomorphisms of $\mathcal{M}_c(X)$ is a *higher order derivation of rank r* if for each α in $\mathbb{N}^r$ we have

$$D_\alpha(\mu * \nu) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D_\beta\mu * D_{\alpha-\beta}\nu, \quad \mu, \nu \in \mathcal{M}_c(X) \tag{7}$$

The *order* of D_α is defined as $|\alpha|$. In particular, D_0 satisfies

$$D_0(\mu * \nu) = D_0\mu * D_0\nu, \tag{8}$$

that is, D_0 is an algebra homomorphism and module homomorphism simultaneously. We call such a mapping a *multiplicative module homomorphism*.

Derivations vs moment functions

Theorem (Ż.F., E. Gselmann and L. Székelyhidi [5])

Let X be a commutative hypergroup and r a positive integer. The family $(D_\alpha)_{\alpha\in\mathbb{N}^r}$ of self-mappings on $\mathcal{M}_c(X)$ is a continuous higher order derivation of order r if and only if there exists a generalized moment function sequence $(\varphi_\alpha)_{\alpha\in\mathbb{N}^r}$ of rank r such that

$$\langle D_\alpha\mu, f\rangle = \int_X f\cdot\varphi_\alpha\,d\mu \tag{9}$$

holds for each μ in $\mathcal{M}_c(X)$, f in $\mathcal{C}(X)$ and α in $\mathbb{N}^r$.

Example

Let $G = \mathbb{R}$, and we consider the functions $\varphi_{k,\lambda}(x) = x^k e^{\lambda x}$ for k in $\mathbb{N}$, where x is in $\mathbb{R}$ and λ is in $\mathbb{C}$. These functions form a generalized moment sequence of rank one as

$$\varphi_{n,\lambda}(x+y) = (x+y)^n e^{\lambda(x+y)} = \sum_{k=0}^{n} \binom{n}{k} x^k e^{\lambda x} y^{n-k} e^{\lambda y}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \varphi_{k,\lambda}(x)\varphi_{n-k,\lambda}(y).$$

This moment function sequence generates the following higher order derivation on the space $\mathcal{M}_c(\mathbb{R})$:

$$\langle D_k\mu, f\rangle = \int_{\mathbb{R}} x^k e^{\lambda x} f(x)\, d\mu(x).$$

Example

We have

$$\begin{aligned}\langle D_k(\psi\mu), f\rangle &= \int_{\mathbb{R}} x^k e^{\lambda x} f(x)\cdot\psi(x)\,d\mu(x)\\ &= \int_{\mathbb{R}} x^k e^{\lambda x} f(x)\,d(\psi\mu)(x) = \langle \psi D_k(\mu), f\rangle\end{aligned}$$

hence D_k is a module homomorphism for each $k = 0, 1, \ldots$. In particular, for $\lambda = 0$

$$\langle D_k\mu, 1\rangle = \int_{\mathbb{R}} x^k\,d\mu(x),$$

which corresponds to the classical moments of the measures.

Remarks and perspectives

- Derivations: a survey by B. Ebanks [3] including the results of E.Gselmann, G.Kiss, C.Vincze [7] among others.
- Investigations: interplay between derivations, moment functions and exponential monomials.
- Examples of derivations on special types of hypergroups.

References

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Thank you very much for your kind attention!