Observability cost of localized and microlocalized data

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A linear transport equation

Consider the following linear transport equation :

$$\partial_t \boldsymbol{u} + (v \cdot \nabla)\boldsymbol{u} + A\boldsymbol{u} = 0, \text{ for } t \in [0, T],$$

where

- $v : (t, x) \in \mathbb{R}^d \mapsto u(t, x) \in \mathbb{R}^d$ is a given regular vector field,
- $A(x)$ is a given regular $N \times N$ matrices field, with $N \in \mathbb{N}^*$.
- $t \in [0, T]$, where the final time $T > 0$ is also given,
- the unknown is the vector field $\boldsymbol{u} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^N$.

Cost of observability

Consider the following linear (vector) transport equation :

$$\partial_t \boldsymbol{u} + (v \cdot \nabla)\boldsymbol{u} + A\boldsymbol{u} = 0, \text{ for } t \in [0, T],$$

Consider a regular field of $n \times N$ matrices B depending on $x \in \mathbb{R}^d$.

We define the observability cost :

$$\mathcal{C}[\boldsymbol{u}_0] := \frac{\|\boldsymbol{u}(T, \cdot)\|_{L^2}^2}{\int_0^T \|B\boldsymbol{u}(t)\|_{L^2}^2 \, \mathrm{d}t},$$

of the initial data $\boldsymbol{u}_0$ at time T by the observation operator B.

Above $\boldsymbol{u}$ is the solution of the transport PDE associated with $\boldsymbol{u}_0$.

Goal : identify the observability cost when the initial energy is localized

Our goal is to identify the observability cost :

$$\mathcal{C}[\boldsymbol{u}_0] := \frac{\|\boldsymbol{u}(T,\cdot)\|_{L^2}^2}{\int_0^T \|B\boldsymbol{u}(t)\|_{L^2}^2 \,\mathrm{d}t},$$

when the energy tensor

$$\boldsymbol{u}_0 \otimes \boldsymbol{u}_0 \quad \text{is close to} \quad (b_0 \otimes b_0)\delta_{x_0},$$

for some $x_0 \in \mathbb{R}^d$ and $b_0 \in \mathbb{R}^N$, for the weak-* topology of the space $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^{N\times N})$ of the matrix-valued measures.

First result on the observability cost of localized data

Theorem

Let $(\boldsymbol{u}_0^n)_n$ *a sequence in* $L^2(\mathbb{R}^d;\mathbb{R}^N)$ *such that*

$$\boldsymbol{u}_0^n \otimes \boldsymbol{u}_0^n \rightharpoonup (b_0 \otimes b_0)\delta_{x_0} \quad \text{weak-* in } \mathcal{M}(\mathbb{R}^d;\mathbb{R}^{N\times N}), \quad \text{as } n \to +\infty.$$

Then,

$$\mathcal{C}[\boldsymbol{u}_0^n]^{-1} \to (b_T^{\mathcal{N}})^* \, G_{x_0}(T)\, b_T^{\mathcal{N}}, \quad \text{as } n \to +\infty.$$

Characteristic flow

For any $x_0 \in \mathbb{R}^d$, the characteristic flow $X_{x_0}(t)$ induced by the vector field v, which appears in

$$\partial_t \boldsymbol{u} + (v \cdot \nabla)\boldsymbol{u} + A\boldsymbol{u} = 0,$$

is defined by :

$$X'_{x_0}(t) = v(X_{x_0}(t)) \text{ for } t \in [0, T] \text{ and } X_{x_0}(0) = x_0.$$

Gramian matrices

We also define the matrices $G_{x_0}(T)$ as the solutions to the following ODE :

$$G'_{x_0}(t) - G_{x_0}(t)A_{x_0}(t) - A^*_{x_0}(t)G_{x_0}(t) = B^*_{x_0}(t)B_{x_0}(t) \text{ for } t \in [0, T] \text{ and } G_{x_0}(0) = 0.$$

with

$$A_{x_0}(t) := A(X_{x_0}(t)) - \frac{1}{2}(\operatorname{div} v)(X_{x_0}(t)) \operatorname{Id}_N \quad \text{and} \quad B_{x_0}(t) := B(X_{x_0}(t)),$$

This is a time-dependent Lyapunov ODE involving the coefficients of the PDE evaluated along the characteristic flow.

Amplitude equation

For $x_0 \in \mathbb{R}^d$ and $b_0 \in \mathbb{R}^N \setminus \{0\}$, we consider $b_{x_0,b_0} \in \mathbb{R}^N \setminus \{0\}$ the solution of the linear ODE :

$$b'_{x_0,b_0} = -A_{x_0} b_{x_0,b_0}, \quad \text{with} \quad b_{x_0,b_0}(0) = b_0.$$

We denote by

$$b_T^{\mathcal{N}} := \frac{b_{x_0,b_0}(T)}{|b_{x_0,b_0}(T)|},$$

its normalization.

First result on the observability cost of localized data

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Then,

$$\mathcal{C}[\boldsymbol{u}_0^n]^{-1} \to (b_T^{\mathcal{N}})^* \, G_{x_0}(T)\, b_T^{\mathcal{N}}, \quad \text{as } n \to +\infty.$$

Idea of the proof

We observe that $u \otimes u$ satisfies :

$$(\partial_t + v \cdot \nabla)(u \otimes u) + A(u \otimes u) + (u \otimes u)A^* = 0,$$

while the "recorder" M implicitly defined by

$$G_{x_0}(t) =: M(t, X_{x_0}(t)),$$

satisfies :

$$(\partial_t + v \cdot \nabla)M + (\operatorname{div} v)M - A^* M - MA = B^* B \quad \text{and} \quad M(0, \cdot) = 0.$$

Idea of the proof

By duality formula, we deduce that

$$\int_{\mathbb{R}^d} M(T,x) : (\boldsymbol{u} \otimes \boldsymbol{u})(T,x)\,\mathrm{d}x = \int_0^T \|B\boldsymbol{u}(t)\|_{L^2}^2\,\mathrm{d}t,$$

which is the observed energy, while the final energy is recast as :

$$\|\boldsymbol{u}(T,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^d} \mathrm{Id} : (\boldsymbol{u} \otimes \boldsymbol{u})(T,x)\,\mathrm{d}x,$$

By passing to the limit, we finally get that

$$\mathcal{C}[\boldsymbol{u}_0^n]^{-1} = \frac{\int_0^T \|B\boldsymbol{u}^n(t)\|_{L^2}^2\,\mathrm{d}t}{\|\boldsymbol{u}^n(T,\cdot)\|_{L^2}^2} \to (b_T^{\mathcal{N}})^*\, G_T\, b_T^{\mathcal{N}}, \quad \text{as } n \to +\infty.$$

Microlocalized data

For more general (linear system of) PDEs, because of the dispersive effects, it is required to consider a phase space analysis.

We then turn to the problem where a sequence of initial data $(\boldsymbol{u}_0^\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2(\mathbb{R}^d;\mathbb{R}^N)$ is such that their semiclassical Wigner matrices satisfy

$$\mathcal{W}^\varepsilon[\boldsymbol{u}_0^\varepsilon] \rightharpoonup (b_0 \otimes b_0)\delta_{x_0,\xi_0} \quad \text{in } \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^{N\times N}), \quad \text{as } \varepsilon \to 0.$$

A glimpse on the result

If

$$\mathcal{W}^{\varepsilon}[\boldsymbol{u}_0^{\varepsilon}] \rightharpoonup (b_0 \otimes b_0)\delta_{x_0,\xi_0} \quad \text{in } \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{N\times N}), \quad \text{as } \varepsilon \to 0,$$

then one may similarly identify the limit of the observability cost $\mathcal{C}[\boldsymbol{u}_0^{\varepsilon}]^{-1}$ in terms of bicharacteristics and microlocalized Gramian matrices.

The analysis is carried in the setting of Weyl pseudodifferential operators and the main assumptions are hyperbolicity and non-crossing of the eigenmodes.

Thanks for your attention !