Atomic decomposition of a subspace of BMO

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Outline

Abstract

The goal of my talk is to introduce a new decomposition of a subspace of BMO. This is a continuation of what I have been doing for other function spaces. Around 1990, Frazier and Jawerth introduced the technique of obtaining non-smooth atoms from wavelet decomposition. This idea was revisited by Grafakos in his text book Modern Harnomic Analysis. This technique together with the reexamination of the atomic decomposition of Hardy spaces with variable exponents brought out a new technique to decompose functions in other spaces such as Triebel-Lizorkin-Morrey spaces. My talk reports an advancement in this direction.

BMO

Recall that a locally integrable function f is BMO if and only if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|}\int_Q \left| f(x) - \frac{1}{|Q|}\int_Q f(y)\mathrm{d}y \right| \mathrm{d}x < \infty.$$

Type of decompositions

- Smooth decomposition (wavelet decomposition, atomic decomposition, quark decomposition, molecular decomposition)

$$\sum_{Q \in \mathcal{D}} \lambda_Q a_Q$$

- Nonsmooth decomposition (for Hardy spaces, Banach lattices and so on)

Today's results can be located as the nonsmooth decomposition, whose ingredient is the smooth decomposition

Elementary notions

We recall the notion of dyadic cubes.

Definition

1. For $j \in \mathbb{Z}$ and $k := (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, let $Q_{jk} := \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_n}{2^j}, \frac{k_n+1}{2^j}\right)$. A dyadic cube is a set of the form Q_{jk} with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. The symbol $\mathcal{D}$ stands for the set of all dyadic cubes.
2. For any $Q \in \mathcal{D}$, let $\ell(Q)$ denote its side length and $j_Q := -\log_2 \ell(Q)$.
3. Sometimes, we identity $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbb{Z},m\in\mathbb{Z}^n}$ with $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ via $\lambda_{\nu,m} = \lambda_Q$ when $Q = Q_{\nu,m}$.

Decreasing rearrangement

Let $L^0(\mathbb{R}^n)$ denote the set of all Lebesgue measurable functions on $\mathbb{R}^n$. Given a function $f \in L^0(\mathbb{R}^n)$, its *distribution function* $d_f : [0, \infty) \to [0, \infty]$ is defined by setting

$$d_f(t) := |\{x \in \mathbb{R}^n \ : \ |f(x)| > t\}|$$

for $t \in [0, \infty)$, and its *decreasing rearrangement* f^* is the function on $[0, \infty)$ defined by setting

$$f^*(t) := \inf \left(\{s \in [0, \infty) \ : \ d_f(s) \le t\} \cup \{\infty\}\right)$$

for $t \in [0, \infty)$.

New atoms (New sequence atoms)

I would like to introduce a new notion of atoms, which is the heart of my talk today.

Definition

A sequence $r := \{r_Q\}_{Q\in\mathcal{D}}$ is said to be an *atom* (sequence atom) centered at a dyadic cube Q_0 if

(i) for any $Q \in \mathcal{D} \setminus \mathcal{D}(Q_0)$, $r_Q = 0$;

(ii) for any $Q \in \mathcal{D}(Q_0)$,

$$\left(\sum_{S\in\mathcal{D}, S\subset Q} |S|^{-1}|r_S\chi_S(\cdot)|^2\right)^* \left(2^{-n-2}|Q|\right) \le 1. \qquad (1)$$

Wavelet (from sequences to distributions)

In order to connect BMO with sequenc spaces we use the wavelets $\{\psi^l\}_{l=1}^{2^n-1}$ defined on $\mathbb{R}^n$. Then, for any $l \in \{1, 2, \dots, 2^n - 1\}$, ψ_l enjoys the following properties for some $N \in \mathbb{N}_0$:

(i) (compact support) each ψ^l is supported on $[-N, N]^n$;

(ii) (regularity) $\psi^l \in C^1(\mathbb{R}^n)$;

(iii) (vanishing moments) $\displaystyle\int_{\mathbb{R}^n} \psi^l(x)\,\mathrm{d}x = 0$.

For a dyadic cube $Q = Q_{jm}$, we write

$$\psi^l_Q(x) = 2^{jn/2}\psi^l(2^j x - m) \quad (x \in \mathbb{R}^n).$$

Lizorkin distributions

Definition

Let

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \ \forall\, \alpha \in \mathbb{N}_0^n, \int_{\mathbb{R}^n} f(x) x^\alpha \, \mathrm{d}x = 0 \right\}.$$

The symbol $\mathcal{S}'_\infty(\mathbb{R}^n)$ denote the dual spaces of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_\infty(\mathbb{R}^n)$ and is called and Lizorkin distributions, respectively. Denote by $\mathcal{P} := \mathcal{P}(\mathbb{R}^n)$ the set of all polynomials. Then we can regard $\mathcal{P}(\mathbb{R}^n)$ as the subset of $\mathcal{S}'(\mathbb{R}^n)$.

It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) := \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

Wave cluster (Elementary unit of my talk)

Definition

Let $Q_0 \in \mathcal{D}$. A distribution $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ is called a *wave cluster* centered at Q_0 if there exists an integer $l \in \{1, 2, \dots, 2^n - 1\}$ and an atom $r := \{r_Q\}_{Q \in \mathcal{D}}$ centered at Q_0 such that

$$f := \sum_{Q \in \mathcal{D}} r_Q \psi^l_Q \tag{2}$$

in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

This definition was taken from the paper by Frazier and Jawerth. [1] But the difference is that we loosen the condition of sequence atoms.

[1] Frazier, M., Jawerth, B.: A discrete transform and decompositions of distribution spaces. J. Funct. Anal. 93, no. 1, 34–170 (1990)

New decomposition of (a subspace of) BMO, the first part

Theorem

Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, and $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of wave clusters centered, respectively, at dyadic cubes $\{Q_j\}_{j\in\mathbb{N}}$ such that $X = \left\| \sum_{j=1}^{\infty} |\lambda_j \chi_{Q_j}| \right\|_{L^\infty} < \infty$. Then there exists a Lizorkin distribution $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{\infty} \lambda_j f_j$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, that $f \in \mathrm{BMO}(\mathbb{R}^n)$, and that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le CX,$$

where $C > 0$ is independent of $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{f_j\}_{j\in\mathbb{N}}$.

The second part

Theorem

Let $p, u \in (0,\infty)$. *Assume that* $f \in \mathrm{BMO}(\mathbb{R}^n)$ *satisfies*

$$\lim_{j\to-\infty}\left(\sum_{S\in\mathcal{D}, S\subset Q_{jk}} |r_S \chi_S(\cdot)|^2\right)^*\left(2^{-n-2}|Q_{jk}|\right) = 0, \text{ for any}$$

$l \in \{1,\ldots,2^n-1\}$ *and* $k \in \{-1,0\}^n$, *where* $r^l := \{\langle f, \psi^l_Q\rangle\}_{Q\in\mathcal{D}}$. *Then there exist a sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ *and a sequence* $\{f_j\}_{j\in\mathbb{N}}$ *of wave clusters centered, respectively, at dyadic cubes* $\{Q_j\}_{j\in\mathbb{N}}$ *such that* $f = \sum_{j=1}^{\infty}\lambda_j f_j$ *in* $\mathcal{S}'_\infty(\mathbb{R}^n)$ *and that*

$$\sup_{Q\in\mathcal{D}} |Q|^{-\frac{1}{p}}\left\|\left(\sum_{j=1}^{\infty}|\lambda_j \chi_{Q_j}|^u\right)^{\frac{1}{u}}\right\|_{L^p(Q)} \le C\|f\|_{\mathrm{BMO}}, \text{ where } C>0.$$

Sparseness

Definition

A collection $\mathcal{F} = \cup_{j=1}^{\infty} \mathcal{F}_j$ of cubes is called a *sparse* family with a level structure if it satisfies the following conditions:

(a) for any $j \in \mathbb{N}$, $\mathcal{F}_j := \{Q_k^j\}_{k \in K_j}$ is a family of disjoint cubes, where K_j is a set of indices;

(b) for any $j \in \mathbb{N}$,

$$\bigcup_{k \in K_{j+1}} Q_k^{j+1} \subset \bigcup_{k \in K_j} Q_k^j;$$

(c) for any $j \in \mathbb{N}$ and $R \in \mathcal{S}_j$,

$$\sum_{k \in K_{j+1}} \left| Q_k^{j+1} \cap R \right| \leq \frac{1}{2} |R|.$$

We may consider the case where $\{Q_j\}_{j=1}^{\infty}$ is sparse.

Outline

1. Main theorem

2. Historical remarks
   - Nonsmooth decomposition of Hardy spaces
   - Nonsmooth decomposition of Triebel–Lizorkin–Morrey spaces

Hardy spaces

We recall the definition of Hardy spaces.

Definition

Let $0 < p < \infty$. The Hardy space $H^p(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which satisfies

$$\|f\|_{H^p} \equiv \left\| \sup_{t>0} |e^{t\Delta} * f| \right\|_{L^p} < \infty.$$

H^p-decomposition, the first part

Theorem

Let $0 < p \le 1$*. Suppose that* $\{f_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ *and* $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$ *satisfy* $\mathrm{supp}(f_j) \subset 3Q_j$ *and*

$$X = \left(\sum_{j=1}^\infty \|f_j\|_{L^\infty}{}^p |Q_j| \right)^{\frac{1}{p}} < \infty.$$

Then $f = \sum_{j=1}^\infty f_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{H^p} \lesssim X.$$

the second part

Theorem

Let $f \in H^p(\mathbb{R}^n)$ *with* $0 < p \le 1$. *Then there exist* $\{f_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ *and* $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$ *such that* $\mathrm{supp}(f_j) \subset 3Q_j$ *for each* j, *that*

$$\left(\sum_{j=1}^\infty \|f_j\|_{L^\infty}{}^p |Q_j| \right)^{\frac{1}{p}} \lesssim \|f\|_{H^p}$$

and that $f = \sum_{j=1}^\infty f_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$.

How did we obtain the decomposition?

We use the Calderòn–Zygmund decomposition of distribution (elements in $\mathcal{S}'(\mathbb{R}^n)$) as is proposed in the Stein book.[2]

[2] E. M. Stein, Harmonic Analysis: Real variable methods, orthogonality, and oscillatory integrals, Princeton University Press

Outline

1. Main theorem
2. Historical remarks
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   - Nonsmooth decomposition of Triebel–Lizorkin–Morrey spaces

Morrey spaces

We recall the definition of the Triebel–Lizorkin–Morrey space $\dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n)$ for $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. For a start, we recall the definition of Morrey spaces.

Definition

Let $0 < q \le p < \infty$. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^0(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} := \sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q)} < \infty.$$

homogeneous Triebel-Lizorkin–Morrey space

Definition

Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\chi_{B(4)\setminus B(2)} \le \varphi \le \chi_{B(8)\setminus B(1)}$. The homogeneous Triebel-Lizorkin–Morrey space $\dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the quantity $\|f\|_{\dot{\mathcal{E}}^s_{pqr}} \equiv \|\{2^{js}\varphi_j(D)f\}_{j\in\mathbb{Z}}\|_{\mathcal{M}^p_q(l^r)}$ is finite, where $\varphi_j(x) \equiv \varphi(2^{-j}x)$, $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$, $\psi(D)f(x) \equiv \mathcal{F}^{-1}\psi * f(x)$, $(x \in \mathbb{R}^n)$ for $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|\{f_j\}_{j\in\mathbb{Z}}\|_{\mathcal{M}^p_q(l^q)}$ stands for the vector-norm of a sequence $\{f_j\}_{j=-\infty}^\infty$ of mesurable functions:

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{\mathcal{M}^p_q(l^r)} \equiv \left\| \left(\sum_{j=-\infty}^\infty |f_j|^q \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

We recall the atomic decomposition of Triebel–Lizorkin–Morrey spaces. This result dates back to 2018. See the paper by Asami and Sawano. [3]

[3] Asami, K., Sawano, Y.: Non-smooth decomposition of homogeneous Triebel–Lizorkin–Morrey spaces, Comment. Math. 58 (2018), no. 1-2, 37–56.

Sequence spaces for $\dot{\mathcal{E}}^s_{pqr}$

Definition

Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. We consider the set of sequences $\{r_Q\}_{Q \in \mathcal{D}} \subset \mathbb{C}$ such that the function

$$g^s_r(\{r_Q\}_{Q \in \mathcal{D}}; x) \equiv \left(\sum_{Q \in \mathcal{D}} (|Q|^{-\frac{s}{n}-\frac{1}{2}} |r_Q| \chi_Q(x))^r \right)^{\frac{1}{r}} \quad (x \in \mathbb{R}^n)$$

is in $\mathcal{M}^p_q(\mathbb{R}^n)$. Let $0 < p < \infty$. For such sequences $r = \{r_Q\}_{Q \in \mathcal{D}}$ we set $\|r\|_{\dot{\mathbf{e}}^s_{pqr}} \equiv \|g^s_r(r)\|_{\mathcal{M}^p_q}$. A sequence $\lambda = \{\lambda_Q\}_{Q \in \mathcal{D}}$ is said to belong to $\dot{\mathbf{e}}^s_{pqr}(\mathbb{R}^n)$ if $\|\lambda\|_{\dot{\mathbf{f}}^s_{pqr}} < \infty$.

Atoms (sequence atoms), classical

We use the decomposition using the distribution as is proposed in the Grafakos book. [4]

Definition

Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. A sequence $\{r_Q\}_{Q \in \mathcal{D}}$ is called an ∞-atom for $\dot{\mathbf{e}}^s_{pqr}(\mathbb{R}^n)$ with cube Q_0 if there exists a dyadic cube Q_0 such that

$$g^s_r(\{r_Q\}_{Q \in \mathcal{D}}; \cdot) \equiv \left(\sum_{Q \in \mathcal{D}} (|Q|^{-\frac{s}{n}-\frac{1}{2}} |r_Q| \chi_Q)^q \right)^{\frac{1}{q}} \le \chi_{Q_0}. \quad (3)$$

[4] L. Grafakos, Modern Fourier Analysis, Graduate texts in mathematics; 250, New York, Springer, 2014.

Wavelet, revisited

We adapt what we defined to the setting above. We use the wavelets $\{\psi^l\}_{l=1}^{2^n-1}$ defined on $\mathbb{R}^n$. Then, for any $l \in \{1, 2, \ldots, 2^n - 1\}$, ψ_l enjoys the following properties for some $N \in \mathbb{N}_0$:

(i) (compact support) each ψ^l is supported on $[-N, N]^n$;

(ii) (regularity) $\psi^l \in C^N(\mathbb{R}^n)$;

(iii) (vanishing moments) $\displaystyle\int_{\mathbb{R}^n} x^\alpha \psi^l(x)\,\mathrm{d}x = 0$ for $|\alpha| \le N$.

For a dyadic cube $Q = Q_{jm}$, we write

$$\psi^l_Q(x) = 2^{jn/2}\psi^l(2^j x - m) \quad (x \in \mathbb{R}^n).$$

Wave cluster (classical)

Definition

Let $Q_0 \in \mathcal{D}$. A distribution $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ is called an ∞-*wave cluster* centered at Q_0 if there exists an integer $l \in \{1, 2, \dots, 2^n - 1\}$ and an ∞-atom $r := \{r_Q\}_{Q \in \mathcal{D}}$ centered at Q_0 such that $f := \sum_{Q \in \mathcal{D}} r_Q \psi^l_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

For this definition of atoms, we can prove the atomic decomposition for parameters p, q, s, u satisfying

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \quad 0 < u \le \min(1, q).$$

$\dot{\mathcal{E}}^s_{pqr}$-decomposition–the first part

Theorem

Let $0 < u \le \min(1, q, r)$. *Assume that each* $f^{l,j}$ *is an* ∞*-wave cluster centered at* $Q_{l,j}$. *Assume that* $\{\lambda^{l,j}\}_{l=1,2,\dots,2^n-1, j=1,2,\dots} \subset \mathbb{C}$ *satisfies*

$$X = \left\| \left(\sum_{j=1}^{\infty} \sum_{l=1}^{2^n-1} |\lambda^{l,j} \chi_{Q_{l,j}}|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \sum_{l=1}^{2^n-1} \lambda^{l,j} f^{l,j}$ *converges in* $\mathcal{S}'_\infty(\mathbb{R}^n)$ *and satisfies* $\|f\|_{\dot{\mathcal{E}}^s_{pqr}} \lesssim X$.

the second part

Theorem

Let $f \in \dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n)$. Then there exist distributions $\{f^{l,j}\}_{l=1,2,\ldots,2^n-1,j=1,2,\ldots}$ and coefficients $\{\lambda^{l,j}\}_{l=1,2,\ldots,2^n-1,j=1,2,\ldots} \subset \mathbb{C}$ such that $f = \sum_{j=1}^{\infty} \sum_{l=1}^{2^n-1} \lambda^{l,j} f^{l,j}$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and that $X \lesssim \|f\|_{\dot{\mathcal{E}}^s_{pqr}}$.

[5]

[5] K. Asami and Y. Sawano, Non-smooth decomposition of homogeneous Triebel–Lizorkin–Morrey spaces, Comment. Math. 58 (2018), no. 1-2, 37–56.

We do not go into the details of applications but we content ourselves with the possibility of obtaining the boundedness of the Marcinkiewicz integrals.

Thank you for your attention!