The Pompeiu problem for locally compact groups

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The classical Pompeiu problem

Let $n \geq 2$ be an integer and let K be a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure. Is $f = 0$ the only continuous function on $\mathbb{R}^n$ that satisfies

$$\int_{\sigma(K)} f(x)\, dx = 0$$

for all rigid motions σ?

The answer is no for circles of positive radius.

A reasonable question is what happens when the set of rigid motions is replaced by a set of translations?

Let G be a locally compact group with Haar measure dx. Let $f \in L^2(G)$ and let K be a closed compact subset of G with positive measure.

Two-sided $L^2(G)$-Pompeiu problem: When is $f = 0$ the only function in $L^2(G)$ that satisfies

$$\int_{gKh} f(x)\, dx = 0$$

for all $g, h \in G$?

If $f = 0$ is the only solution to this equation for a given K, then we shall say that K is a $L^2(G)$-Pompeiu set.

If K is a closed, normal compact subgroup of G, then K is not a $L^2(G)$-Pompeiu set.

Suppose G is unimodular, second countable and type I.

Let $f \in L^1(G) \cap L^2(G)$ and let μ be the Plancherel measure on $\hat{G}$. The Fourier transform $f \mapsto \hat{f}$ maps $L^1(G) \cap L^2(G)$ into $\int \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}} \, d\mu(\pi)$. Let χ_K be the characterestic function on K. Set

$$Z(\chi_K) = \{\pi \in \hat{G} \mid \widehat{\chi_k}(\pi) = 0\}.$$

Theorem: Let K be a closed compact subset of G. K is a $L^2(G)$-Pompeiu set if and only if $\mu(Z(\chi_K)) = 0$.

We shall say that G is a $L^2(G)$-Pompeiu group if every closed compact subset of G of positive measure is a $L^2(G)$-Pompeiu set. A group G is compact-free if it does not contain nontrivial compact subgroups

If G is compact-free abelian, then $\mu(Z(\chi_K)) = 0$ for all closed compact subsets K of G. Benjamin Weiss (1968).

Hence, if G is compact-free abelian, G is a $L^2(G)$-Pompeiu group

Further results: Let G be a locally compact Nilpotent group. Then G is a $L^2(G)$-Pompeiu group if and only if G is compact-free.

(Probably true) If G is type I, solvable, compact-free Lie group, then G is a $L^2(G)$-Pompeiu group.

Discrete Groups

When is $f = 0$ the only function in $\ell^2(G)$ that satisfies

$$\sum_{x \in gKh} f(x) = 0$$

for all $g, h \in G$?

A finite set K in G is a $\ell^2(G)$-Pompeiu set if $f = 0$ is the only solution to the above equation. A $\ell^2(G)$-Pompeiu group is a group for which every nonempty finite subset of G is a $\ell^2(G)$-Pompeiu set.

For $g \in G$, the left translation of f by g is given by $L_g f(x) = f(gx)$.

Some results

Proposition: Let K be a nonempty finite subset of G and let χ_K be the characteristic function on K. Then K is a $\ell^2(G)$-Pompeiu set if and only if $f = 0$ is the only function in $\ell^2(G)$ that satisfies $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$.

Theorem: Let G be a discrete group. Then G is a $\ell^2(G)$-Pompeiu group if and only if G does not contain a nontrivial finite normal subgroup.

Denote by $\mathcal{F}(G)$ the set of functions on G. (Since G is discrete this is the set of continuous functions on G.)

Theorem: Let G be a discrete group and suppose K is a finite subset of G. Let I be the ideal in $\mathbb{C}G$ that is generated by χ_K. Then K is a $\mathcal{F}(G)$-Pompeiu set if and only if $I = \mathbb{C}G$.

Can we use this theorem to find $\mathcal{F}(G)$-Pompeiu groups?

Define a ring homomorphism $\epsilon : \mathbb{C}G \to \mathbb{C}$ by $\epsilon(f) = \sum_{g \in G} f(g)$ for $f \in \mathbb{C}G$. The augmentation ideal of $\mathbb{C}G$, denoted by $\omega(\mathbb{C}G)$, is the kernel of ϵ. Every group ring has at least three ideals, the trivial ideals and $\omega(\mathbb{C}G)$.

The augmentation ideal is the only proper ideal in the group ring of algebraically closed groups and universal groups! (Bonvallet, Hartley, Passman, Smith, 1976)

Thus for algebraically closed groups and universal groups G the ideal generated by χ_K, where K is a nonempty finite subset G, must be all of $\mathbb{C}G$ since $\epsilon(\chi_K) = |K|$. Hence, algebraically closed groups and universal groups are $\mathcal{F}(G)$-Pompeiu groups.

Thank you