Tiling and weak tiling in $\mathbb{Z}_p^d$

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Harmonic and Spectral Analysis 2023

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Tiles and spectral sets

Let G be a finite Abelian group (for this talk), and $A \subset G$.

A tiles with B, if the translated copies $A + b$ ($b \in B$) cover G unilaterally. In notation, $A + B = G$. In functional notation, this is equivalent to $1_A * 1_B = 1_G$.

A is spectral with spectrum $S \subset \widehat{G}$, if the characters $\gamma \in S$ restricted to A form an orthogonal basis in $L^2(A)$.

Fuglede's conjecture (1974)

Ω is spectral if and only if it is a tile.

We aim to study one of the few remaining open cases of the conjecture: $G = \mathbb{Z}_p^3$ (and discuss $G = \mathbb{Z}_p^d$ briefly, for other values of d).

Weak tiling and weak pd-tiling

Weak tiling is a natural generalization of tiling: A tiles its complement A^c *weakly*, if there exists a nonnegative function $h_0 \geq 0$, such that $1_A * h_0 = 1_{A^c}$.

(This was introduced by Nir Lev and myself in connection with Fuglede's conjecture for convex bodies.)

In this case, adding a unit mass at 0, we get a weak tiling of G. That is, with $h = \delta_0 + h_0$, we have $1_A * h = 1_G$.

In the sequel, we will require furthermore that h be positive definite.

Weak pd-tiling of G

We say that A pd-tiles G weakly if there exists $h : G \to \mathbb{R}$, $h \geq 0$, $h(0) = 1$, $\hat{h} \geq 0$, such that $1_A * h = 1_G$.

Weak pd-tiling induced by tiling and spectrality

Lemma

Assume A tiles G, or A is spectral in G. Then A pd-tiles G weakly.

Indeed, if $1_A * 1_B = 1_G$, then $1_A * (\frac{1}{|B|} 1_B * 1_{-B}) = 1_G$ is a weak pd-tiling.

Similarly, if S is a spectrum of A, it is not hard to prove that $1_A * (\frac{1}{|S|^2} |\widehat{1_S}|^2 = 1_G$ is a weak pd-tiling.

So, weak pd-tiling is a common generalization of tiling and spectrality.

Pd-flat groups

Assume that a finite abelian group G has the property that whenever a set A pd-tiles G weakly then A tiles G properly. Then we call G *pd-flat*.

In any pd-flat group the spectral $\rightarrow$ tile direction of Fuglede's conjecture holds.

$\mathbb{Z}_p^d$ is pd-flat or not?

It is known by results of Ferguson, Sothanaphan and Mattheus (2019-2020) that there exist for all $d \geq 4$ and all odd primes p there exist spectral sets in $\mathbb{Z}_p^d$ which do not tile.

Therefore, $\mathbb{Z}_p^d$ is not pd-flat for $d \geq 4$.

For $d = 1, 2$ we have managed to prove that $\mathbb{Z}_p^d$ is pd-flat. However, the question remains open for $d = 3$.

Therefore, we still do not know whether the spectral $\rightarrow$ tile direction of Fuglede's conjecture holds in $\mathbb{Z}_p^3$. Very disappointing...

Functional pd-tiling and ray-type functions

Functional pd-tiling

Let $f, h : G \to \mathbb{R}$ be nonnegative functions such that $f(0) = h(0) = 1$, $\widehat{f} \geq 0, \widehat{h} \geq 0$. We say that the pair (f, h) is a *functional pd-tiling* of G if $f * h = 1_G$.

Ray-type functions

We say that a function $f : \mathbb{Z}_p^d \to \mathbb{R}$ is *ray-type*, if f is constant on punctured lines).

For any set $A \subset \mathbb{Z}_p^d$ the zeros of the Fourier transform $\widehat{1}_A$ consist of punctured lines through the origin. This fact leads to a natural averaging procedure:

Let $1_A * h_1 = 1_G$ be a weak pd-tiling. Let $f_k = \frac{1}{|A|} 1_{kA} * 1_{-kA}$, and $f = \frac{1}{p-1} \sum_{k=1}^{p-1} f_k$. Then f is ray-type, and $f * h_1$ is a functional pd-tiling.

Complementary 4-tuples

So, we have arrived at f being ray-type, and $f * h_1$ being a functional pd-tiling.

We can also average h_1 as $h_k(x) = h_1(kx)$, and $h = \frac{1}{p-1}\sum_{k=1}^{p-1} h_k$.

Then h is also ray-type, and $f * h$ is a functional pd-tiling with the properties:

$f, h, \widehat{f}, \widehat{h}$ are all ray-type functions,
$f \geq 0$, $h \geq 0$, $\widehat{f} \geq 0$, $\widehat{h} \geq 0$,
$f(0) = 1$, $h(0) = 1$, $\widehat{f}(0) = |A|$, $\widehat{h}(0) = |G|/|A|$,
$f * h = 1_G$, $\widehat{f} * \widehat{h} = |G| 1_{\widehat{G}}$,
$supp\ f \cap supp\ h = \{0\}$, $supp\ \widehat{f} \cap supp\ \widehat{h} = \{0\}$.

We call $f, h, \widehat{f}, \widehat{h}$ a complementary 4-tuple.

Application to $d = 1, 2$

Theorem

$\mathbb{Z}_p^d$ is pd-flat for $d = 1, 2$.

Proof by characterizing all complementary 4-tuples.

The case of $d = 3$

Let $G = \mathbb{Z}_p^3$, and assume $1_A * h_1 = 1_G$ is a weak pd-tiling. Let $f, h, \widehat{f}, \widehat{h}$ be the induced complementary 4-tuple after averaging.

After analyzing complementary 4-tuples in $\mathbb{Z}_p^3$, the following conclusion holds:

Either A tiles or

supp f, *supp* h, *supp* $\widehat{f}$, *supp* $\widehat{h}$ are all blocking sets (in the terminology of Romanos).

We cannot exclude this case.