A linear programming approach to Fuglede's conjecture in $\mathbb{Z}_p^3$

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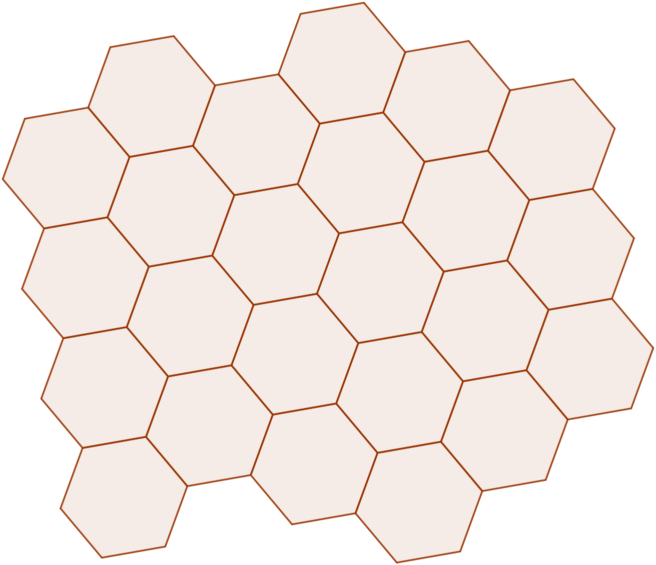
Tiling

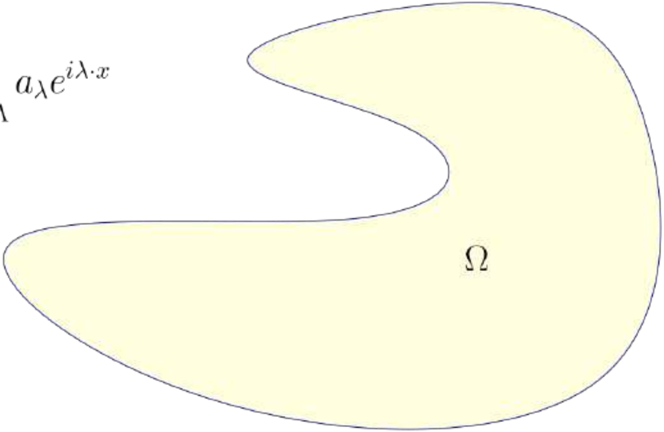


Spectrality

$f \in L^2(\Omega)$

$$\int_\Omega e^{i(\lambda-\lambda')\cdot x}\mathrm{d}x = 0, \quad \lambda \neq \lambda'$$

$$f(x) = \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda\cdot x}$$



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Fuglede's conjecture is still open for $d = 1, 2$. It is true for convex bodies (Lev, Matolcsi '22).

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Results

1. If G has at least 10 generators, then $S \not\Rightarrow T$ (Ferguson, Sophanathan, '22)
2. If G has odd order and at least 4 generators, then $S \not\Rightarrow T$ (Aten et al. '17)
3. If G has at most 2 generators, we only have positive results so far (many authors...)
4. If $G = \mathbb{Z}_p^3$, then $T \Rightarrow S$ (Aten et al.)

$\hat{G} = \{\xi : G \to \mathbb{C} : \xi(x+y) = \xi(x)\xi(y), \forall x, y \in G\}$. Since G finite, $\xi(x)$ is a root of unity.

Discrete Fourier Analysis

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Fourier transform

For $f : G \to \mathbb{C}$ define $\hat{f} : \hat{G} \to \mathbb{C}$ as

$$\mathbf{F}f(\xi) = \hat{f}(\xi) = \sum_{x \in G} f(x)\xi(-x) = \langle f, \xi \rangle$$

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Inverse Fourier transform: $f(x) = \frac{1}{|G|} \sum_{\xi \in \hat{G}} \hat{f}(\xi)\xi(x)$.

Convolution: $f * g(x) = \sum_{y \in G} f(x-y)g(y)$. $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Parseval: $\mathbf{U} = \frac{1}{\sqrt{|G|}}\mathbf{F}$ is unitary: $|G| \sum_{x \in G} |f(x)|^2 = \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2$.

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Restricting inner products on $A \subset G$:

$$\langle f, g \rangle_A = \sum_{x \in A} f(x)\overline{g(x)} = \langle f|_A, g|_A \rangle.$$

$\xi, \psi \in \hat{G}$ are orthogonal on A if $\langle \xi, \psi \rangle_A = 0$ (Notation: $\xi \perp \psi$.)

Let $E \subset G$, such that $0 \in E$ and $E = -E$ (*forbidden set*). We seek to maximize $|B|$ such that $(B - B) \cap E = \{0\}$.

Delsarte's method

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Witness function

A function $h : G \to \mathbb{R}$ is called a *witness function* with respect to E if

(a) h is even and $h(x) \leq 0$, $\forall x \in G \setminus E$.

(b) $\hat{h} \geq 0$, $\hat{h}(0) > 0$.

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Theorem (Delsarte '72)

With B, E, h as above, it holds

$$|B| \leq |G| \cdot \frac{h(0)}{\hat{h}(0)}.$$

Spectrum

$B \subset \hat{G}$ is a set of orthogonal characters of $A \subset G$, if for every $\xi \neq \psi$, $\xi, \psi \in B$ we have

$$0 = \langle \xi, \psi \rangle_A = \sum_{x \in A} (\xi \psi^{-1})(x) = \hat{\mathbf{1}}_A(\xi \psi^{-1})$$

or equivalently,

$$B - B \subset Z(\hat{\mathbf{1}}_A) \cup \{0\}.$$

If in addition $|B| = |A|$, then B is a spectrum of A (it always holds $|B| \leq |A|$).

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Remark

$h = \widehat{\mathbf{1}_A * \mathbf{1}_{-A}} = |\hat{\mathbf{1}}_A|^2$ *is a witness function for* E *which achieves equality, i. e.* $|G| \cdot h(0)/\hat{h}(0) = |A|$.

$G = \mathbb{Z}_p^3$

Fix an isomorphism $G \cong \hat{G}$, under the map $x \mapsto \xi_x$, where $\xi_x(y) = \zeta_p^{\langle x,y \rangle}$, with $\zeta_p = e^{2\pi i/p}$ and

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

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Balanced (or ray-type) functions

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Balanced witness function

If h is a witness function for a union of lines E, then g is also a witness function for E, where

$$g(x) = \frac{1}{p-1} \sum_{\lambda \in \mathbb{Z}_p^*} h(\lambda x)$$

is in addition a balanced function.

Passing to $\mathbf{P}\mathbb{F}_p^2$

$[x : y : z] = [\lambda x : \lambda y : \lambda z]$ for $\lambda \neq 0$. The affine plane is included in $\mathbf{P}\mathbb{F}_p^2$ via the map $(x, y) \mapsto [x : y : 1]$; for $z = 0$ we get the line at infinity.

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If S is a union of punctured lines in $\mathbb{Z}_p^3$, then the corresponding set of points in $\mathbf{P}\mathbb{F}_p^2$ is denoted by $\tilde{S}$.

Fourier analysis on the finite projective plane

If L is a line through O, then:

$$\hat{\mathbf{1}}_O = \mathbf{1}_{\mathbb{Z}_p^3}, \quad \hat{\mathbf{1}}_L = p\mathbf{1}_{L^\perp}, \quad \hat{\mathbf{1}}_{L^*} = p\mathbf{1}_{L^\perp} - \mathbf{1}_{\mathbb{Z}_p^3}$$

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Functions on projective plane

For $f : \mathbb{Z}_p^3 \to \mathbb{C}$ balanced, define $\tilde{f} : \mathbf{P}\mathbb{F}_p^2 \cup \{O\} \to \mathbb{C}$ as $\tilde{f}([x : y : z]) = f(x, y, z)$, $\tilde{f}(O) = f(O)$. The Fourier transform is defined to satisfy $\hat{\tilde{f}} = \tilde{\hat{f}}$.

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Abusing notation, we write $O = [0 : 0 : 0]$. For $P = [x : y : z] \in \mathbf{P}\mathbb{F}_p^2$ define

$$P^\perp = \left\{ Q = [u : v : w] \in \mathbf{P}\mathbb{F}_p^2 : xu + yv + zw = 0 \right\}.$$

$$\hat{\delta}_P = p\delta_{P^\perp} + p\delta_O - \mathbf{1}, \quad \hat{\delta}_O = \mathbf{1}.$$

Blocking sets

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Facts:

- If Z is a blocking set, then so is Z^c.
- If $A \subset \mathbb{Z}_p^3$ is spectral, then $p \mid |A|$. If $|A| = p$ or p^2, then it tiles. If $|A| > p^2$, then $A = \mathbb{Z}_p^3$. Otherwise, $|A| = pk$, with $1 < k < p$ and

$$\widetilde{Z(\hat{\mathbf{1}}_A)} = \left\{ [x : y : z] \in \mathbf{P}\mathbb{F}_p^2 : \hat{\mathbf{1}}_A(x, y, z) = 0 \right\} = Z^c$$

  is a blocking set, and so is $Z = \widetilde{\mathrm{supp}\hat{\mathbf{1}}_A}$ (Fallon, Mayeli, Villano '19).
- Let Z' be the smallest blocking set such that $Z' \subset Z$. Then (Bruen, Thas '77),

$$|Z'| \le p\sqrt{p} + 1.$$

Finding the witness function

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h is a witness function for $E = \mathbb{Z}_p^3 \setminus Z(\hat{\mathbf{1}}_A)$:

- The first condition ($h \leq 0$ *outside* E) is satisfied, as $\operatorname{supp}\tilde{h} \subset Z \cup \{O\}$.
- The second condition (*positivity of* $\hat{h}$) is satisfied:

$$\hat{\tilde{h}} = p\left(\sum_{P \in Z'} \delta_{P^\perp} + |Z'|\delta_O - \mathbf{1}\right),$$

so that for $Q \in \mathbf{P}\mathbb{F}_p^2$

$$\hat{\tilde{h}}(Q) = p\left(\sum_{P \in Z'} \delta_{P^\perp}(Q) - 1\right) = p\left(\sum_{P \in Z'} \delta_{Q^\perp}(P) - 1\right) = p(|Z' \cap Q^\perp| - 1) \geq 0$$

and $\hat{\tilde{h}}(O) = p(|Z'| - 1) > 0.$

Finding the witness function

Suppose that $B \subset \hat{G}$ is a (maximal) set of pairwise orthogonal characters on A. Delsarte's method with witness function h gives us

$$|B| \leq |G| \cdot \frac{h(0)}{\hat{h}(0)} = p^3 \cdot \frac{\tilde{h}(O)}{\hat{\tilde{h}}(O)} = p^3 \cdot \frac{|Z'| - p}{p(|Z'| - 1)} = p^2 \left(1 - \frac{p-1}{|Z'| - 1}\right).$$

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Remark

- *It takes care about $\sqrt{p}$ multiples of p between $2p$ and $(p-1)p$.*
- *Previously known only for $k = p-2$ or $p-1$ (Fallon, Mayeli, Villano '19).*

Work in progress - Open questions

1. Could Z' be smaller? At any rate, not smaller than $\frac{3}{2}(p+1)$; in this case, if

$$p \cdot \frac{p^2+5p}{3p+1} < |A| < p^2,$$

then A is not spectral, using the same method

2. Could Z intersect every line in more than one points? Z either intersects every line at 3 points at least, or the points of A are distributed in k parallel planes, each having exactly p points of A.

3. If Z is a t-blocking set (i. e. it intersects every line at $\geq t$ points), then

$$\tilde{h} = \delta_{Z'} + (|Z'| - tp)\delta_O$$

is a witness function with respect to $E = \mathbb{Z}_p^3 \setminus Z(\hat{\mathbf{1}}_A)$, where Z' is a minimal t-blocking subset of Z. Applying Delsarte's method on h and using bounds on the size of minimal 3-blocking sets, yield that A is not spectral for $\approx \sqrt{3p}$ values of k.

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Thank you!