L^p-L^q Fourier multipliers on hypergroups

(Joint work with Michael Ruzhansky)

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Boundedness of Fourier multipliers

- The operator $A : C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is called a Fourier multiplier with symbol $\sigma_A : \mathbb{R}^n \to \mathbb{C}$ if it satisfies

$$\mathcal{F}_{\mathbb{R}^n}(Af)(\xi) = \sigma_A(\xi)(\mathcal{F}_{\mathbb{R}^n}f)(\xi) \quad \xi \in \mathbb{R}^n, f \in C_c^\infty(\mathbb{R}^n).$$

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- $\mathcal{L}$: non-negative self-adjoint operator on $L^2(X)$ with spectrum contained in $[0, \infty)$. Then, for any bounded Borel function $\varphi : [0, \infty) \to \mathbb{C}$, using the spectral theorem, we define spectral multiplier of $\mathcal{L}$ (formally) by

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- L^2-boundedness: Easy !
- L^p-boundedness Fourier multipliers and spectral multipliers: Hörmander-Mihlin type condition & Marcinkiewicz type condition Hörmander, Clerc and Stein, Taylor, Coifman and Weiss, Fefferman, Seeger, Anker, Cowling and Sikora, Guilini, Meda, Mauceri, Christ, Alexopoulos, Müller, Ricci, Thiele, Hebisch, Thangavelu, Ruzhansky and many others.

L^p-L^q Fourier multipliers on $\mathbb{R}^n$

- We are interested in the L^p-L^q boundedness of Fourier and spectral multipliers for $1 < p \leq q < \infty$.

[1] L. Hörmander, Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140 (1960).

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[4] V. Kumar and M. Ruzhansky, L^p-L^q boundedness of (k, a)-Fourier multipliers with applications to nonlinear equations, *Int. Math. Res. Not. (IMRN))*, 2, (2023) 1073-1093.

[5] R. Akylzhanov and M. Ruzhansky, $L^p - L^q$ multipliers on locally compact groups, *J. Func. Anal.*, 278(3) (2019), DOI: https://doi.org/10.1016/j.jfa.2019.108324

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- Hörmander (1960)[1] : For the range $1 < p \leq 2 \leq q < \infty$,

$$\sup_{s>0} s \left(\int_{\{\xi \in \mathbb{R}^n : |\sigma_A(\xi)| \geq s\}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}} < \infty$$

$\Rightarrow A$ has a bounded extension from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

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- This result has been extended to several settings, such as compact Lie groups, compact quantum groups[2], compact hypergroups[3], the generalized Dunkl-Fourier transform[4], eigenfunction expansions, locally compact groups [5], and commutative hypergroups [6].

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COMMUTATIVE HYPERGROUPS

A *commutative hypergroup*[7] [8] is a non empty locally compact Hausdorff space H with a weakly continuous, associative convolution $*$ on the Banach space $M(H)$ of all bounded regular Borel measures on H such that $(M(H), *)$ becomes a Banach algebra and the following properties hold:

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(i) For any $x, y \in H$, the convolution $\delta_x * \delta_y$ is a probability measure with compact support, where δ_x is the point mass measure at x. Also, the mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous from $H \times H$ to the space $\mathcal{C}(H)$ of all nonempty compact subsets of H equipped with the Michael (Vietoris) topology.

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(ii) There exists a unique element $e \in H$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for every $x \in H$.

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(iii) There is a homeomophism $x \mapsto \check{x}$ on H of order two which induces an involution on $M(H)$ where $\check{\mu}(E) = \mu(\check{E})$ for any Borel set E, and $e \in \text{supp}(\delta_x * \delta_y)$ if and only if $x = \check{y}$.

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Examples: Every locally compact abelian group is a trivial example of a commutative hypergroup. Other important examples include double closet spaces, Bessel-Kingman hypergroups, Jacobi hypergroups and Chébli-Trimèche hypergroups.

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Basic of Fourier analysis on commutative hypergroups

- A *left Haar measure* λ on H is a non-zero positive Radon measure λ such that

$$\int_H f(x*y)d\lambda(y) = \int_H f(y)\, d\lambda(y) \quad (\forall x \in H, f \in C_c(H)),$$

where we used the notation $f(x*y) = (\delta_x * \delta_y)(f)$.

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- The dual space of a commutative hypergroup H is defined by

$$\widehat{H} = \{\chi \in C^b(H) : \chi \neq 0,\, \chi(\check{x}) = \overline{\chi(x)} \text{ and } \chi(x*y) = \chi(x)\chi(y)\ \forall\, x, y \in H\}.$$

  The elements of $\widehat{H}$ are called *characters* of H. We equip $\widehat{H}$ also with the compact-open topology so that H becomes a locally compact Hausdorff space. Note that $\widehat{H}$ need not possess a hypergroup structure.

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The elements of $\widehat{H}$ are called *characters* of H. We equip $\widehat{H}$ also with the compact-open topology so that H becomes a locally compact Hausdorff space. Note that $\widehat{H}$ need not possess a hypergroup structure.

- The Fourier transform of $f \in L^1(H, \lambda)$ is defined by

$$\widehat{f}(\chi) = \int_H f(x)\overline{\chi(x)}\,d\lambda(x), \ \ \forall\, \chi \in \widehat{H}. \tag{1}$$

Two basic (in)equalities for Fourier transform on commutative hypergroups

- There exists a unique positive Borel measure π on $\widehat{H}$ such that

$$\int_H |f(x)|^2 \, d\lambda(x) = \int_{\widehat{H}} |\widehat{f}(\chi)|^2 \, d\pi(\chi) \;\; \forall f \in L^2(H,\lambda) \cap L^1(H,\lambda).$$

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- In fact, the Fourier transform extends to a unitary operator from $L^2(H,\lambda)$ onto $L^2(\widehat{H},\pi)$.
- The support of π, denoted $\mathcal{S}$, need not be equal to $\widehat{H}$

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Theorem 1 (Hausdorff-Young inequality[9])

Let p, p' be such that $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $f \in L^2(H, d\lambda) \cap L^1(H, d\lambda)$ we have the inequality

$$\|\widehat{f}\|_{L^{p'}(\widehat{H},d\pi)} \le \|f\|_{L^p(H,d\lambda)}.$$

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The Paley inequality

- If a positive function φ on $\mathbb{R}^n$ satisfies $|\{\xi \in \mathbb{R}^n : \varphi(\xi) \geq t\}| \leq \frac{C}{t}$ for $t > 0$ then

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^p \varphi(\xi)^{2-p} \, d\xi \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p \leq 2.$$

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Theorem 2

Let H be a commutative hypergroup equipped with a Haar measure λ and let $\widehat{H}$ be the dual of H equipped with measure π. Suppose that ψ is a positive function on $\widehat{H}$ satisfying the condition

$$M_\psi := \sup_{t>0} t \int_{\substack{\chi \in \widehat{H} \\ \psi(\chi) \geq t}} d\pi(\chi) < \infty. \tag{2}$$

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Then for $f \in L^p(H, d\lambda)$, $1 < p \leq 2$, we have

$$\left(\int_{\widehat{H}} |\widehat{f}(\chi)|^p\, \psi(\chi)^{2-p} d\pi(\chi)\right)^{\frac{1}{p}} \lesssim M_\psi^{\frac{2-p}{p}} \|f\|_{L^p(H,d\lambda)}. \tag{3}$$

THE HAUSDORFF-YOUNG-PALEY INEQUALITY

Theorem 3

Let H be a commutative hypergroup equipped with a Haar measure λ and let $\widehat{H}$ be the dual of H equipped with measure π. Let $1 < p \leq 2$, and let $1 < p \leq b \leq p' < \infty$, where $p' = \frac{p}{p-1}$. If $\psi(\chi)$ is a positive function on $\widehat{H}$ such that

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is finite,

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is finite, then for every $f \in L^p(H, d\lambda)$ we have

$$\left(\int_{\widehat{H}} \left(|\widehat{f}(\chi)| \psi(\chi)^{\frac{1}{b} - \frac{1}{p'}} \right)^b d\pi(\chi) \right)^{\frac{1}{b}} \lesssim M_\psi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(H, d\lambda)}. \tag{5}$$

L^p-L^q Fourier multipliers on commutative hypergroups

- For a function $h \in L^\infty(\widehat{H}, d\pi)$, define the operator T_h as

$$\widehat{T_h f}(\chi) = h(\chi)\widehat{f}(\chi), \quad \chi \in \widehat{H},$$

for all f belonging to a suitable function space on $\widehat{H}$. The operator T_h is called the Fourier multiplier on H with symbol h.

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  for all f belonging to a suitable function space on $\widehat{H}$. The operator T_h is called the Fourier multiplier on H with symbol h.
- It is clear that T_h is a bounded operator on $L^2(H, d\lambda)$ by the Plancherel theorem.

Theorem 4

Let $1 < p \leq 2 \leq q < \infty$ and let H be a commutative hypergroup. Suppose that T_h is a Fourier multiplier with symbol h. Then we have

$$\|T_h\|_{L^p(H,d\lambda) \to L^q(H,d\lambda)} \lesssim \sup_{s>0} s \left[\int_{\{\chi \in \widehat{H} : |h(\chi)| \geq s\}} d\pi(\chi) \right]^{\frac{1}{p} - \frac{1}{q}}.$$

Chébli-Trimèche hypergroups

Now, we will focus on a class of "one dimensional hypergroups" on $\mathbb{R}_+$ called *Chébli-Trimèche hypergroups*[10] [11] [12] with the convolution structure related to the following second-order differential operator

$$L = L_{A,x} := -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)}\frac{d}{dx}, \tag{6}$$

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[12] K. Trimèche, Transformation intègrale de Weyl et thèorème de Paley-Wiener associés à un opérateur différentiel singulier sur $(0, \infty)$. (French) [Weyl integral transform and Paley-Wiener theorem associated with a singular differential operator on $(0, \infty)$], *J. Math. Pures Appl. (9)* 60(1) (1981) 51-98.

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where the function A, called a *Chébli-Trimèche function*, is continuous on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+^* := (0, \infty)$ and satisfies the following properties:

(i) $A(0) = 0$ and A is positive on $\mathbb{R}_+^*$.

(ii) A is an increasing function and $A(x) \to \infty$ as $x \to \infty$.

(iii) $\frac{A'}{A}$ is a decreasing C^∞- function on $\mathbb{R}_+^*$ and hence $\rho := \frac{1}{2} \lim_{x\to\infty} \frac{A'(x)}{A(x)} \geq 0$ exists.

(iv) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighbourhood of 0, where $\alpha > -\frac{1}{2}$ and B is an odd C^∞-function on $\mathbb{R}$.

[10] H. Chébli, Generalized translation operators and convolution semi-groups (English), *Theory of Potential and Harmonic Analysis (Journées Soc.Mat.France, Institute for Advanced Mathematical Research, Strasbourg, 1973)* Reading Notes in Math., Vol. 404, Springer, Berlin (1974) 35-59.

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Chébli-Trimèche hypergroups

- A hypergroup $(\mathbb{R}_+, *)$ is called a Chébli-Trimèche hypergroup if there exists a Chébli-Trimèche function A such that for any real-valued C^∞-function f on $\mathbb{R}_+$, i.e., the restriction of an even non-negative C^∞-function on $\mathbb{R}$, the generalised translation $u(x,y) = T_x f(y) := \int_0^\infty f(t)\, d(\delta_x * \delta_y)(t), \quad y \in \mathbb{R}_+$ is the solution of the following Cauchy problem:

$$(L_{A,x} - L_{A,y})u(x,y) = 0,$$
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- The growth of $(\mathbb{R}_+, *(A))$ is determined by the number $\rho := \frac{1}{2} \lim\limits_{x \to \infty} \frac{A'(x)}{A(x)}$. We say that $(\mathbb{R}_+, *(A))$ is of *exponential growth* if and only if $\rho > 0$. Otherwise we say that the hypergroup is of *subexponential growth* which also includes the polynomial growth.

Some examples

The class of Chébli-Trimèche hypergroups contains many important classes of hypergroups. We will discuss three of them here.

(i) If $A(x) := x^{2\alpha+1}$ with $2\alpha \in \mathbb{N}$ and $\alpha > -\frac{1}{2}$ then $L_{A,x}$ is the radial part of the Laplace operator on the Euclidean space and $(\mathbb{R}_+, *(A))$ is a Bessel-Kingman hypergroup.

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(iii) If A is the density function on the simply connected harmonic manifold X of purely exponential volume growth then $L_{A,x}$ is the radial part of the Laplace-Beltrami operator on X and $(\mathbb{R}_+, *(A))$ is the “radial hypergroup" of X.

Fourier analysis on Chébli-Trimèche hypergroups

- The Haar measure m on $(\mathbb{R}_+, *(A))$ is given by $m(x) := A(x)\,dx$, where dx is the usual Lebesgue measure on $\mathbb{R}_+$.

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- For the Chébli-Trimèche hypergroup $(\mathbb{R}_+, *(A))$, the multiplicative functions on $(\mathbb{R}_+, *(A))$ are given by the eigenfunctions of the operator $L := L_{A,x}$ defined in (6). For any $\lambda \in \mathbb{C}$, the equation
$$Lu = (\lambda^2 + \rho^2)u \tag{7}$$
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- The dual space $\widehat{\mathbb{R}}_+$ of $(\mathbb{R}_+, *(A))$ is described by $\{\phi_\lambda : \lambda \in [0, \infty) \cup [0, i\rho]\}$ which can be identified with the parameter set $\mathbb{R}_+ \cup [0, i\rho]$.

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- We define the Fourier transform $\widehat{f}$ of $f \in L^1(\mathbb{R}_+, Adx)$ at a point $\lambda \in \widehat{\mathbb{R}}_+$ by
$$\widehat{f}(\lambda) := \int_0^\infty f(x)\,\phi_\lambda(x)\,A(x)dx.$$

Levitan-Plancherel identity

Theorem 5

There exists a unique non-negative measure π *on* $\widehat{\mathbb{R}}_+$ *with support* $[\rho^2, \infty)$ *such that the Fourier transform induces an isometric isomorphism from* $L^2(\mathbb{R}_+, Adx)$ *onto* $L^2(\widehat{\mathbb{R}}_+, \pi)$ *and for any* $f \in L^1(\mathbb{R}_+, Adx) \cap L^2(\mathbb{R}_+, Adx)$,

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- The Plancherel measure $d\pi := C_0|c(\lambda)|^{-2}d\lambda$, where C_0 is a positive constant and the function c satisfies the following: there exist positive constants C_1, C_2 and K such that for any $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \leq 0$,

$$C_1|\lambda|^{a+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2|\lambda|^{a+\frac{1}{2}} \quad \text{for} \quad |\lambda| \leq K,$$

$$C_1|\lambda|^{\alpha+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2|\lambda|^{\alpha+\frac{1}{2}} \quad \text{for} \quad |\lambda| > K.$$

Spectral multipliers for generalised Laplacian

- L^p-boundedness of Fourier multipliers on Chébli-Trimèche hypergroups were established by Bloom and Xu [13].

[13] W. R. Bloom and Z. Xu, Fourier multipliers for L^p on Chébli-Trimèche hypergroups, *Proc. London Math. Soc. (3)* 80(3) (2000) 643-664.

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Theorem 6

Let $1 < p \leq 2 \leq q < \infty$ and Let T_h be a Fourier multiplier with symbol h. Then we have

$$\|T_h\|_{L^p(\mathbb{R}_+,Adx)\to L^q(\mathbb{R}_+,Adx)} \lesssim \sup_{s>0} s \left[\int_{\{\lambda\in\mathbb{R}_+:|h(\lambda)|\geq s\}} |c(\lambda)|^{-2}\, d\lambda\right]^{\frac{1}{p}-\frac{1}{q}}.$$

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Theorem 7

Let $1 < p \leq 2 \leq q < \infty$ and let φ be a monotonically decreasing continuous function on $[\rho^2, \infty)$ such that $\lim_{u\to\infty} \varphi(u) = 0$. Then we have

$$\|\varphi(L)\|_{\mathrm{op}} \lesssim \sup_{u>\rho^2} \varphi(u) \begin{cases} (u-\rho^2)^{(\alpha+1)(\frac{1}{p}-\frac{1}{q})} & \text{if} \quad (u-\rho^2)^{\frac{1}{2}} \leq K, \\ \left[K^{2\alpha+2} - K^{2\alpha+2} + (u-\rho^2)^{(\alpha+1)}\right]^{\frac{1}{p}-\frac{1}{q}} & \text{if} \quad (u-\rho^2)^{\frac{1}{2}} > K, \end{cases} \tag{8}$$

where K is a constant appearing in the estimate of the c-function and $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm from $L^p(\mathbb{R}_+, Adx)$ to $L^q(\mathbb{R}_+, Adx)$.

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Corollary 1

Let L be the generalised radial Laplacian. For any $1 < p \leq 2 \leq q < \infty$ there exists a positive constant $C = C_{\alpha,a,p,q,K}$ such that

$$\|e^{-tL}\|_{\text{op}} \lesssim \begin{cases} t^{-2(\alpha+1)(\frac{1}{p}-\frac{1}{q})} & \text{if} \quad 0 < t < \frac{\alpha+1}{K}\left(\frac{1}{p}-\frac{1}{q}\right) \\ e^{-t\rho^2}\, e^{-\frac{(a+1)^2}{t}\left(\frac{1}{p}-\frac{1}{q}\right)^2} t^{-2(a+1)(\frac{1}{p}-\frac{1}{q})} & \text{if} \quad t \geq \frac{a+1}{K}\left(\frac{1}{p}-\frac{1}{q}\right), \end{cases}$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm from $L^p(\mathbb{R}_+, Adx)$ to $L^q(\mathbb{R}_+, Adx)$.

Thank you for your attention!