Connectifying Counterexamples

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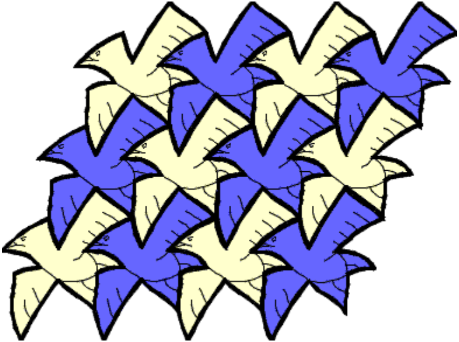
Aperiodic Tiling

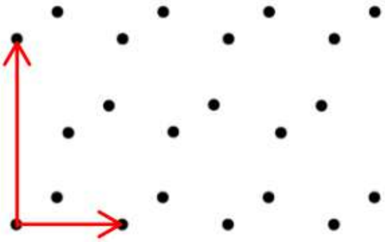
- Suppose $A \subseteq \mathbb{Z}^d$ is finite.
- A **tiles** if there is $T \subseteq \mathbb{Z}^d$ such that $A \oplus T = \mathbb{Z}^d$.



- T is **periodic** if its period group

$$P = \left\{ g \in \mathbb{Z}^d : \; T + g = T \right\}$$

spans $\mathbb{Z}^d$.



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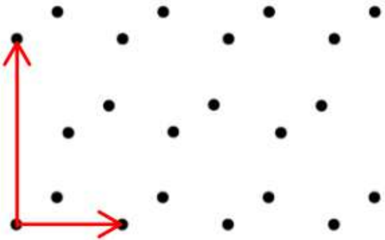
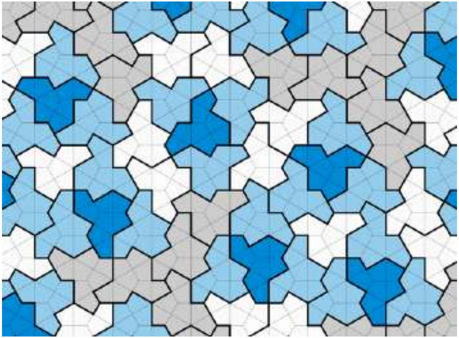


- $A \subseteq \mathbb{Z}^d$ is an **aperiodic tile** if it tiles but only with translates T which are not periodic.

Aperiodic Tiling

- If we allow rotations and reflections then such aperiodic tiles are known to exist, even in the plane.



- Greenfeld and Tao (2022):

Theorem

If d is sufficiently large then there is a finite $A \subseteq \mathbb{Z}^d$ that tiles $\mathbb{Z}^d$ by translations but only non-periodically.

Aperiodic Tiling

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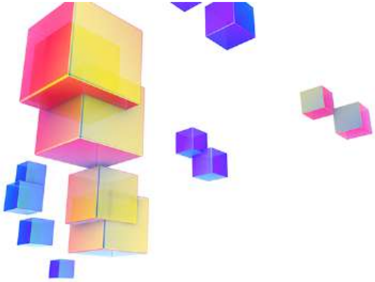

- Thus they disproved the **Periodic Tiling Conjecture** in high dimension.

- A finite set $A \subseteq \mathbb{Z}^d$ is **spectral** if there exists $\Lambda \subseteq \mathbb{T}^d$ such that

$$\left\{ e_\lambda(n) = e^{2\pi i \lambda n} : \lambda \in \Lambda \right\}$$

is orthogonal and $|\Lambda| = |A|$ (completeness).

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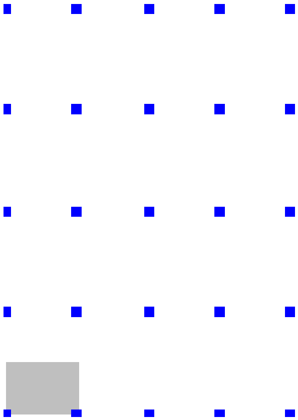
Conjecture (Fuglede, 1974)

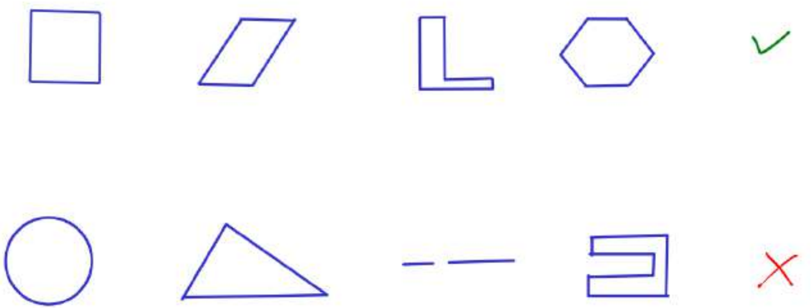
A is a translational tile $\Longleftrightarrow$ *A is spectral.*

A rectangle and one of its spectra.



Tiles and spectral sets

More examples of spectral and non-spectral sets



Disproof of the Fuglede Conjecture

- Tao (2003): Constructed $A \subseteq \mathbb{Z}^5$ which is spectral but not a tile.

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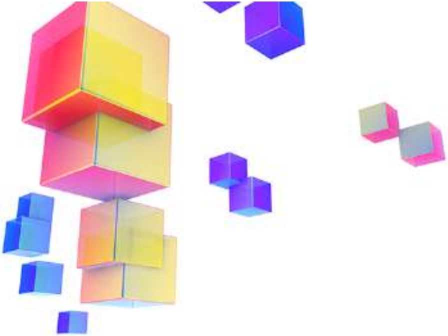
- Due to the work of many others (K., Matolcsi, Farkas, Révész, Móra) both directions are now known to fail for $d \geq 3$.
- Again, all examples are highly dispersed.



- Demanding some properties from a class of sets may affect tiling and spectrality.
- Lev and Matolcsi (2019): The Fuglede conjecture is true for the class of convex bodies.
- Beauquier, Nivat, Kenyon (1991-92): The Periodic Tiling Conjecture is true for topological disks in the plane.

Results

- Demanding some properties from a class of sets may affect tiling and spectrality.
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- Question: Could it be that demanding **connectivity** changes the game in aperiodicity and spectrality?

RESULTS

- Demanding some properties from a class of sets may affect tiling and spectrality.
- Lev and Matolcsi (2019): The Fuglede conjecture is true for the class of convex bodies.
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- Question: Could it be that demanding **connectivity** changes the game in aperiodicity and spectrality?
- Answer is NO if one is willing to increase the dimension.
- Larger dimension $\implies$ more freedom to construct examples.

Aperiodicity

Theorem

If

$$A \subseteq \mathbb{Z}^d$$

*tiles $\mathbb{Z}^d$ aperiodically only then we can construct a **connected***

$$A' \subseteq \mathbb{Z}^{d+2}$$

that does the same.

Spectral sets that do not tile

THEOREM

If

$$A \subseteq \mathbb{Z}^d$$

is spectral but does not tile then we can construct a ***connected***

$$A' \subseteq \mathbb{Z}^{d+2}$$

that does the same.

Non-spectral tiles

THEOREM

If

$$A \subseteq \mathbb{Z}^d$$

tiles but is not spectral then we can construct a ***connected***

$$A' \subseteq \mathbb{Z}^{d'}$$

that does the same, for some $d' > d$.

- The difference $d' - d$ depends on the set A.

Aperiodicity preserving operation

Theorem

Let F be a finite subset of $\mathbb{Z}^d$. Define the finite set

$$X = \{(v_j, s_j) : j = 0, 1, \ldots, n-1\} \subseteq \mathbb{R}^{d+k}$$

where $v_0, \ldots, v_{n-1} \in \mathbb{Z}^d$ are arbitrary and $s_0, \ldots, s_{n-1}$ are n distinct points in $\mathbb{Z}^k$ such that

$$S = \{s_j \colon j = 0, 1, \ldots, n-1\}$$

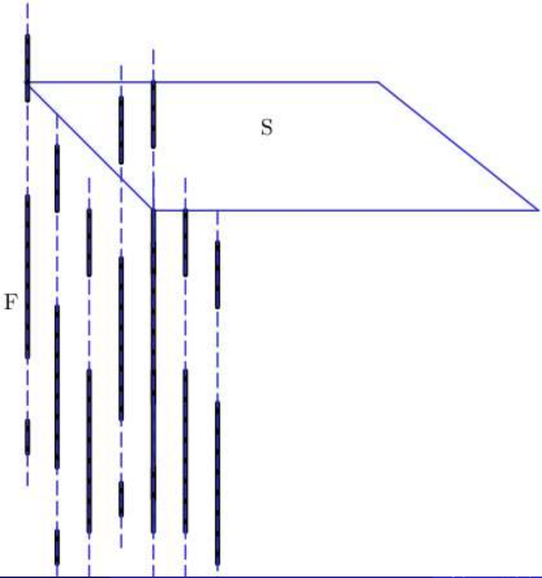
tiles $\mathbb{Z}^k$ by translations.

Let $F' = \left(F \times \{0\}^k\right) \oplus X$.

Then F' is an aperiodic tile in $\mathbb{Z}^{d+k}$ if F is an aperiodic tile of $\mathbb{Z}^d$.

Connected aperiodic tiles

- Imagine copies of F "hanging" from S at various depths



Proof of the Theorem

- Suppose $F' = \left(F \times \{0\}^k\right) \oplus X$ tiles periodically

$$F' \oplus A' = \mathbb{Z}^{d+k}, \text{ where } A' = A' + G' \text{ and } G' \text{ a lattice.}$$

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- Define the lattice $G = G' \cap (\mathbb{Z}^d \times \{0\}^k)$
- And the set $A = (A' + X) \cap (\mathbb{Z}^d \times \{0\}^k)$

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$$F' \oplus A' = \mathbb{Z}^{d+k}, \text{ where } A' = A' + G' \text{ and } G' \text{ a lattice.}$$

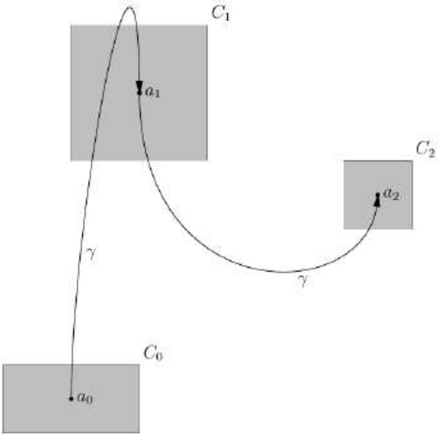
- Define the lattice $G = G' \cap (\mathbb{Z}^d \times \{0\}^k)$
- And the set $A = (A' + X) \cap (\mathbb{Z}^d \times \{0\}^k)$
- Then $A = A + G$ (A is G-periodic) and

$$\begin{aligned}\mathbb{Z}^d \times \{0\}^k &= \mathbb{Z}^{d+k} \cap (\mathbb{Z}^d \times \{0\}^k) \\ &= (F' \oplus A') \cap (\mathbb{Z}^d \times \{0\}^k) \\ &= (F \times \{0\}^k) \oplus (X + A') \cap (\mathbb{Z}^d \times \{0\}^k) \\ &= (F \oplus A) \times \{0\}^k \text{ (a periodic tiling).}\end{aligned}$$

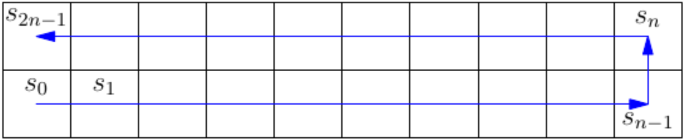
Connected aperiodic tiles

- Let C_i be the connected components of F.
- Pick points $a_i \in C_i$. Connect with a path

$$\gamma : v_0, v_1, \ldots, v_{n-1}.$$



- S is a rectangle in $\mathbb{Z}^2$.



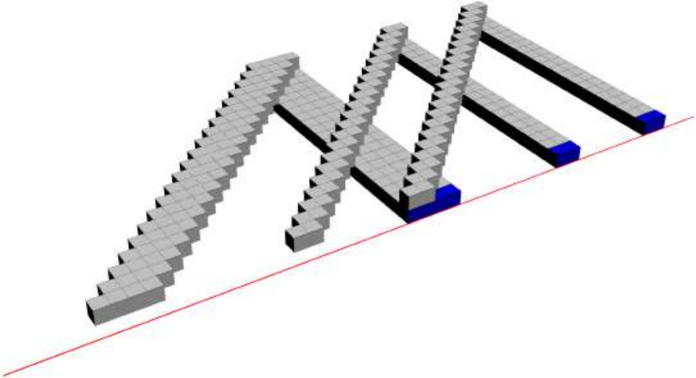
- Define $X \subseteq \mathbb{Z}^{d+2}$ by

$$X = \{X_0, X_1, \ldots, X_{2n-1}\}$$
$$= \{(0, s_0), (0, s_1), \ldots, (0, s_{n-1}), (v_0, s_n), (v_1, s_{n+1}), \ldots, (v_{n-1}, s_{2n-1})\}.$$

LEMMA

The set X is connected in $\mathbb{Z}^{d+2}$.

The folded bridge – A connected aperiodic tile



Theorem

The set $F' = \left(F \times \{0\}^2\right) \oplus X$ *is connected in* $\mathbb{Z}^{d+2}$.

Connected spectral non-tiles

Theorem

Let Ω be a bounded, measurable set in $\mathbb{R}^d$. Define the finite set

$$X = \{(v_j, s_j) : j = 0, 1, \ldots, n-1\} \subseteq \mathbb{R}^{d+k}$$

where $v_0, \ldots, v_{n-1} \in \mathbb{R}^d$ and $s_0, \ldots, s_{n-1}$ are n distinct points in $\mathbb{Z}^k$ such that

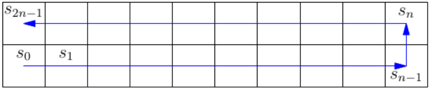
$$S = \{s_j : j = 0, 1, \ldots, n-1\}$$

tiles $\mathbb{Z}^k$ by translations.
Let $\Omega' = \left(\Omega \times [0,1]^k\right) \oplus X$. Then

1. *Ω' tiles $\mathbb{R}^{d+1}$ by translations if and only if Ω tiles $\mathbb{R}^d$ by translations.*
2. *If $\Omega \subset \mathbb{R}^d$ and $S + [0,1]^k \subset \mathbb{R}^k$ are spectral, then Ω' is spectral in $\mathbb{R}^{d+k}$.*

Connected spectral non-tiles

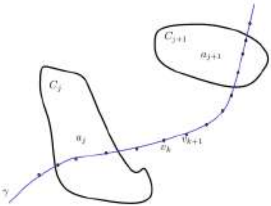
- S is a rectangle, as before.



- We define the set $X \subseteq \mathbb{Z}^{d+2}$ as follows

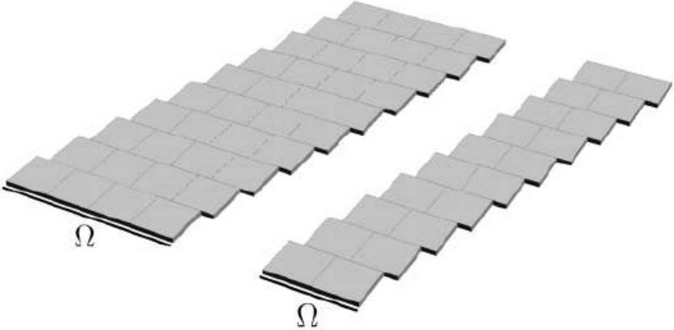
$$\begin{aligned} X &= \{X_0, X_1, \ldots, X_{2n-1}\} \\ &= \{(0, s_0), (0, s_1), \ldots, (0, s_{n-1}), (v_0, s_n), (v_1, s_{n+1}), \ldots, (v_{n-1}, s_{2n-1})\}. \end{aligned}$$

Now the path $v_0, v_1, \ldots, v_{n-1}$ is such that $|v_j - v_{j+1}|$ is very small.
This ensures connectivity of Ω'.



A stacking of $\Omega \subseteq \mathbb{R}^d$

$$\Omega' = \Omega \times [0,1] + \{0, u, 2u, \ldots, (n-1)u\}, \quad u = (v, 1), v \in \mathbb{R}^d$$



Connected non-spectral tiles

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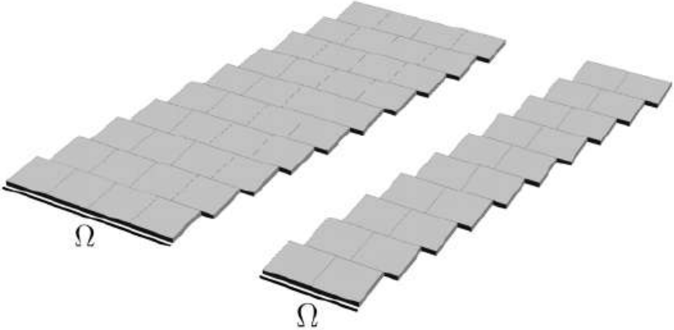

Theorem

$\Omega' \subseteq \mathbb{R}^{d+1}$ is spectral (tile) $\iff$ $\Omega \subseteq \mathbb{R}^d$ is spectral (tile).

Connected non-spectral tiles

- $\mathbf{1}_{\Omega'} = \mathbf{1}_{\Omega\times[0,1]} * (\delta_0 + \delta_u + \cdots + \delta_{(n-1)u})$

$$\widehat{\mathbf{1}_{\Omega'}}(\xi) = \widehat{\mathbf{1}_{\Omega}}(\xi_1, \ldots, \xi_d)\widehat{\mathbf{1}_{[0,1]}}(\xi_{d+1}) \left(\sum_{j=0}^{n-1} e^{2\pi ij(u\cdot\xi)}\right)$$
$$= \widehat{\mathbf{1}_{\Omega}}(\xi_1, \ldots, \xi_d)\widehat{\mathbf{1}_{[0,1]}}(\xi_{d+1}) \frac{1-e^{2\pi in(u\cdot\xi)}}{1-e^{2\pi i(u\cdot\xi)}}, \quad \text{if } u\cdot\xi \notin \mathbb{Z} \qquad (1)$$

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$$= \widehat{\mathbf{1}_\Omega}(\xi_1,\ldots,\xi_d)\widehat{\mathbf{1}_{[0,1]}}(\xi_{d+1})\frac{1-e^{2\pi i n(u\cdot\xi)}}{1-e^{2\pi i(u\cdot\xi)}}, \quad \text{if } u\cdot\xi\notin\mathbb{Z} \qquad (1)$$

- Define the subgroup of $\mathbb{R}^{d+1}$

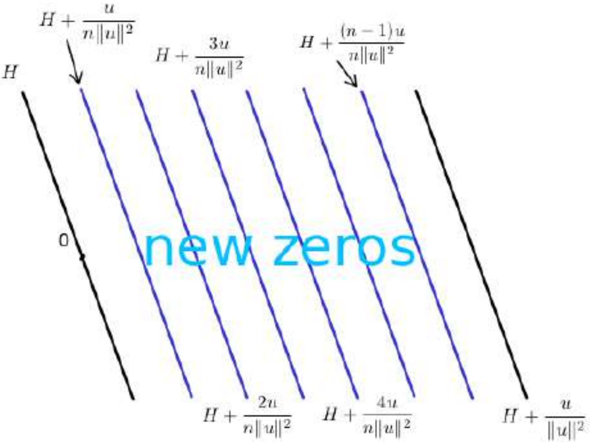
$$G = \left\{\xi = (\xi_1,\xi_2,\ldots,\xi_{d+1}) : u\cdot\xi \in \frac{1}{n}\mathbb{Z}\right\}$$

and its subgroup of index n

$$H = \{\xi = (\xi_1,\xi_2,\ldots,\xi_{d+1}) : u\cdot\xi \in \mathbb{Z}\}.$$

- Zeros of $\widehat{\mathbf{1}_{\Omega'}}$ are those of $\widehat{\mathbf{1}_{\Omega\times[0,1]}}$ plus the set

$$D = \left(H + \frac{u}{n\|u\|^2}\right) \cup \left(H + \frac{2u}{n\|u\|^2}\right) \cup \ldots \cup \left(H + \frac{(n-1)u}{n\|u\|^2}\right).$$



Connected non-spectral tiles

- Suppose $\Lambda' \subseteq \mathbb{R}^{d+1}$ is a spectrum of Ω'.
- Then $\mathrm{dens}\,\Lambda' = |\Omega'| = n|\Omega| = n|\Omega \times [0,1]|$

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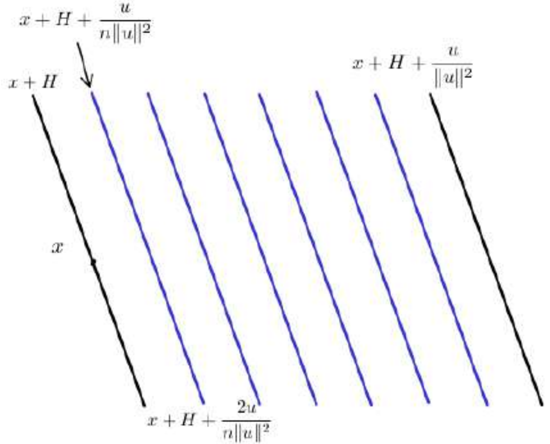
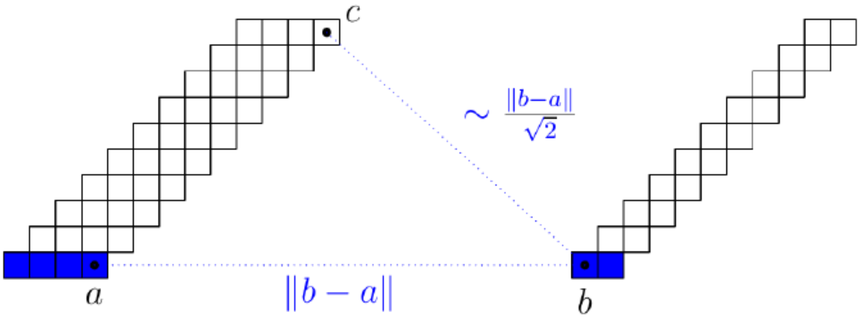
- Then $\operatorname{dens}\Lambda' = |\Omega'| = n|\Omega| = n|\Omega \times [0,1]|$
- We select $\Lambda \subseteq \Lambda'$ with $\Lambda - \Lambda \cap D = \emptyset$ and $\operatorname{dens}\Lambda \geq |\Omega| = |\Omega \times [0,1]|$.
- $\Lambda - \Lambda \subseteq \left\{\widehat{\mathbf{1}_{\Omega\times[0,1]}}\right\} \cup \{0\}$ and Λ is a spectrum of $\Omega \times [0,1]$ (K. 2016).
- $\Omega \times [0,1]$ is spectral $\implies$ Ω is spectral (Greenfeld and Lev 2016).

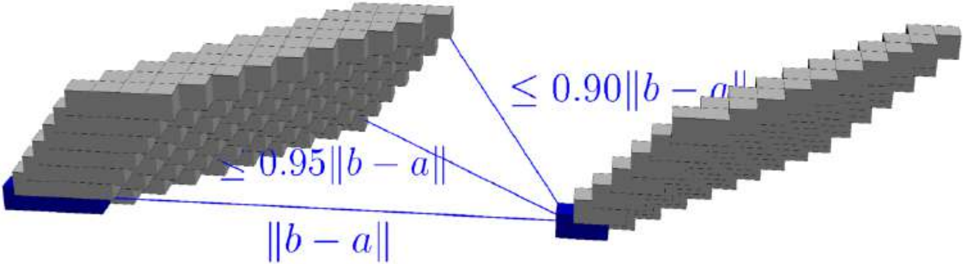
- How to select Λ from Λ'. Look at cosets $x + G$.
- On each coset select the coset of H most populated with λ's.
- Selection Λ has density $\geq \frac{1}{n}\mathrm{dens}\,\Lambda' = |\Omega|$.
- $\Lambda - \Lambda$ are all "old zeros".



A stacking of Ω reduces intercomponent distance

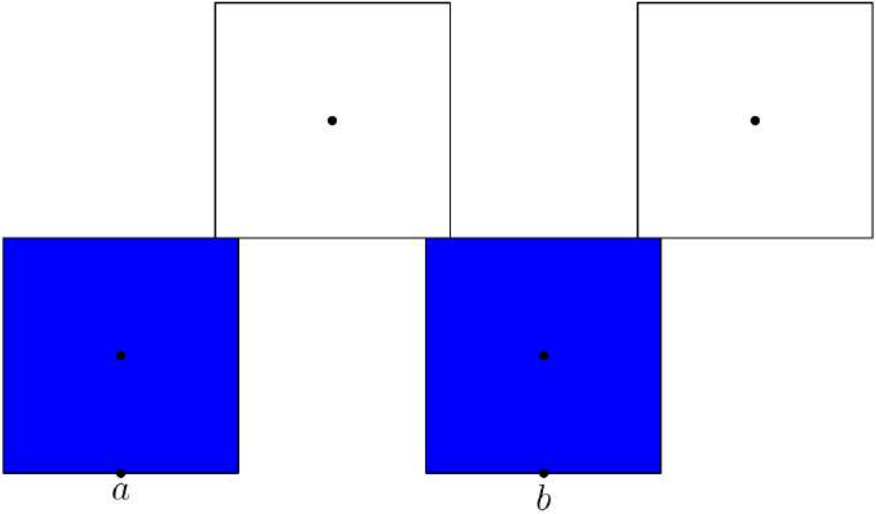


A stacking can be repeated



- Two components can come very close.
- The number of steps (increase in dimension) depends on the components.

Eventually the components get connected



Thank you