Exact signal recovery and restriction theory

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- Fourier Inversion says that we can reconstruct (or recover) a signal completely using its Fourier its Fourier transforms:

$$f(x) = \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- For practical applications, the basic question is, can we *still* recover f **exactly** from its discrete Fourier transforms if some values

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- The answer turns out to be $\boxed{YES}$ if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Uncertainty Principle (UP).

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- (Plancherel)

$$\sum_{m \in \mathbb{Z}_N^d} |\widehat{f}(m)|^2 = N^{-d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

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$$E(x) = \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m)\widehat{E}(m)$$

- $$= \sum_{m \notin S} \chi(x \cdot m)\widehat{E}(m) + \sum_{m \in S} \chi(x \cdot m)\widehat{E}(m) = I(x) + II(x).$$

An elementary point of view: Cauchy-Schwarz

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- Extending the sum in S over the sum in $\mathbb{Z}_N^d$ and applying Plancherel, we see that this expression is bounded by

$$|S|^{\frac{1}{2}} \cdot N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}}.$$

An elementary point of view: rounding

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$$|S|^{\frac{1}{2}} \cdot N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} < \frac{1}{2},$$
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- But what happens if we consider general signals?

Donoho-Stark point of view

- Suppose that $h : \mathbb{Z}_N \to \mathbb{C}$ has N_t non-zero values, and its Fourier transform $\widehat{h}$ has N_w non-zero entries. Then the classical Uncertainty Principle says that

$$|supp(h)| \cdot |supp(\hat{h})| = N_t \cdot N_w \geq N.$$

Notation: We use N_t for support in 'time' and N_w for support in 'frequency'.

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- Suppose that $f : \mathbb{Z}_N \to \mathbb{C}$ is supported in $E \subset \mathbb{Z}_N$, with the frequencies in $S \subset \mathbb{Z}_N$ unobserved.

- What does it mean that we can recover f uniquely? It means that if there exists a signal $g : \mathbb{Z}_N \to \mathbb{C}$ such that g has N_t non-zero entries, and $\widehat{f}(m) = \widehat{g}(m)$ for $m \notin S$, then f must be equal to g.

Uncertainty Principle (UP) $\rightarrow$ Unique Recovery

- To see this, let $h = f - g$. It is clear that $\widehat{h}$ has at most N_w non-zero entries, and h has at most $2N_t$ non-zero entries.

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- Therefore, if
$$N_t \cdot N_w < \frac{N}{2},$$
we must have $h = 0$, and hence the recovery is *unique*.

Uncertainty Principle in real life!

An elementary proof of the (finite) Uncertainty Principle

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- By Fourier Inversion,

$$f(x) = \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m) \quad \forall\, x \in E.$$

- By Cauchy-Schwarz, Plancherel, and the fact that f is supported on E,

$$|f(x)|^2 \leq |S| \cdot \sum_{m \in S} \left|\widehat{f}(m)\right|^2$$

$$= |S| \cdot \sum_{m \in \mathbb{Z}_N^d} \left|\widehat{f}(m)\right|^2 = |S| \cdot N^{-d} \cdot \sum_{x \in E} |f(x)|^2.$$

Conclusion of the proof of UP

- Summing both sides over E and dividing by $\sum_{x \in E} |f(x)|^2$, we get

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$$|E| \cdot |S| \geq N^d, \quad \text{(the classical UP).}$$

- An immediate question that arises is whether this inequality can be improved.

- In general, we cannot do better, but in most cases we can. This, in essence, is the main thrust of this talk.

Restriction theory enters the picture

- We say that $S \subset \mathbb{Z}_N^d$ satisfies the (p, q) restriction estimate $(1 \leq p \leq q)$ with uniform constant $C_{p,q} > 0$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \leq C_{p,q} N^{-d} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

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Theorem

[Uncertainty Principle via Restriction Theory – I & M, 2023] Suppose that $f, \widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$, with f supported in $E \subset \mathbb{Z}_N^d$, and $\widehat{f}$ supported in $S \subset \mathbb{Z}_N^d$. Suppose S satisfies the (p,q) restriction estimate with norm $C_{p,q}$. Then

$$|E|^{\frac{1}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}}.$$

From Restriction to Exact Recovery

Corollary

[Exact Recovery via Restriction Theory – I & M, 2023] Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ *with support* $supp(f) = E$. *Let* r *be another signal with support of the same size such that* $\widehat{r}(m) = \widehat{f}(m)$ *for* $m \notin S$, *and* 0 *otherwise. Suppose* $S \subset \mathbb{Z}_N^d$ *satisfies the* (p, q) *restriction estimate with uniform constant* $C_{p,q}$. *Then* f *can be reconstructed from* r *uniquely if*

$$|E|^{\frac{1}{p}} \cdot |S| < \frac{N^d}{2^{\frac{1}{p}} C_{p,q}}.$$

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- This raises a rather important question of when we can expect $S \subset \mathbb{Z}_N^d$ to satisfy the (p, q) restriction estimate with norm $C_{p,q}$? This is where we now turn our attention.

From additive energy to restriction

Theorem (I &M, 2023)

Let $S \subset \mathbb{Z}_N^d$ *with the property that*

$$|S| = \Lambda_{size} N^{\frac{d}{2}},$$

and

$$|\{(x, y, x', y') \in U : x + y = x' + y'\}| \leq \Lambda_{energy} \cdot |U|^2$$

for every $U \subset S$.

Then S *satisfies* $(\frac{4}{3}, 2)$ *restriction with* $C_{p,q} = \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}}$, *i.e*

$$\left(\frac{1}{|S|}\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}} \leq \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-d}\left(\sum_{x\in\mathbb{Z}_N^d}|f(x)|^{\frac{4}{3}}\right)^{\frac{3}{4}}.$$

Randomness $\rightarrow$ Additive Energy $\rightarrow$ Restriction Estimate

- Suppose that $S \subset \mathbb{Z}_N^d$ is chosen randomly with respect to the uniform distribution. It follows from the results by Dubickas, Schoen, Silva and Sarka (2013) that

$$\mathbb{E}(|\{(x,y,x',y') \in S^4 : x+y = x'+y'\}|) \leq (5+\Lambda_{size}^2)|S|^2.$$

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- We deduce that if S is chosen randomly, and $|S| = \Lambda_{size} N^{\frac{d}{2}}$ with $\Lambda_{size} > 1$, then for any $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|}\sum_{m\in S} |\widehat{f}(m)|^2\right)^{\frac{1}{2}} \leq \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-d}\left(\sum_{x\in\mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}}\right)^{\frac{3}{4}}$$

$$\leq N^{-d}\left(\sum_{x\in\mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}}\right).$$

Summary of the talk