Archimedes Angle Measure: The Fundamental Theorem of Trigonometry

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Café Conversation

The following conversation took place in a café,
the kind frequented by *Jorge Luis Borges*:

The place was noisy and memories are fuzzy.

Maybe the details were different.

Maybe the conversation never took place at all...

Anyhow, we'll try to recount what we remember

and interpret what we recount.

Dialogue

S: " What is the Fundamental Theorem of Trigonometry?"

S: Is it

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad ?$$

T: " That's famous and precedes the Fundamental Theorem, but it's a corollary."

S: "How about

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,? \quad "$$

T: "Getting warmer. This too is a corollary,
less distant in time from the Fundamental Theorem."

T: " ... you know, *counting rotations* is a primordial idea, ... makes us aware of the *integers* ;
Archimedes' measure of partial rotations leads to *completeness* of the *real numbers*."
T: " ... eh, it's less well known, but Archimedes' angle measure also unearths the notion of *complex numbers*, paving the way to the Fundamental Theorem of Trigonometry.

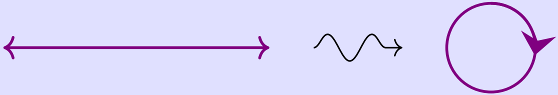
The Fundamental Theorem

Theorem (Fundamental Theorem of Trigonometry)

There's an additive surjective continuous map of the real line to the unit circle,, unique up to scaling.

Additive means:

addition of reals corresponds to addition of angles manifested by rotations on the unit circle.



Angle Measure–Arc Length

Intro books may gloss over arc length, *angle measure*. Appeal to intuition:

Take a string, pull it taut, measure with ruler. Try it:

Sometimes we need to measure *small* strings. Small strings can fray: 

Some say that we don't need angle measure–just trig functions of angle measure.

But we need to understand X to handle the limit $\lim_{x \to 0} \dfrac{\sin(x)}{x}$.

Typically limit handled by inscribed/circumscribed area arguments, without analytic foundation or development.

Some books do more, e.g.,:

O. Toeplitz, *The Calculus, A Genetic Approach*, U. Chicago (1963).

A Circular Argument

That most calculus presentations of $\lim_{x \to 0} \frac{\sin(x)}{x}$ involve circular reasoning has been noticed more than once:

Fred Richman (1993) *A Circular Argument*, Coll. Math J., 24:2, 160-162

Options, remedies listed:

Legend: (***) is Archimedes' Convexity Axiom; (*) is $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

- Continue being circular. After all, a proof is just a completely convincing argument. The students accept the area formula, so why not use it?
- Use Archimedes' $(***)$, explicitly or tacitly. Maybe it's obvious—Johnson and Kiokemeister evidently thought that it was.
- Postulate a suitable form of $(*)$. In the form of the definition of arc length, this is the current view of a logical development. Moreover Gillman says that deriving the area formula from this "is what so pleased the students."
- Define angle measure using area, as Apostol does, thus postponing the whole question of arc length.

We will advocate using the Archimedes Convexity Axiom.

What is An Angle?

WHAT IS AN ANGLE?*

HANS ZASSENHAUS, McGill University

1. Introduction. In this paper I plan to present to you my ideas on how the measurement of angles may be suitably introduced in our time. I am, however, fully aware that some of you rather desire an "angle" from which to look at what is taking place in modern algebra. I ask you, therefore, to accept with patience my diversions from the geometrical angle to the philosophical one and back to the subject matter of this paper.

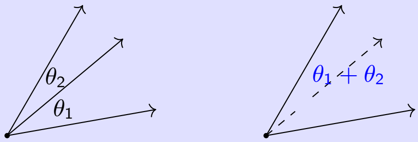
As possible concepts on which the definition of an angle may be based there come to mind:

1) A pair of intersecting straight lines,
2) A pair of rays from the same origin,
3) An open convex part of the plane bounded by a pair of rays with the same origin, called *angular space*,
4) A rotation,
5) A circular arc,
6) An isosceles triangle.

Monthly, 1954.

Expectations from Angle Measure

Whatever angle measure should be–it should be *additive*:



When angles stack, measures should add.

Archimedes introduced angle measure θ in estimating $3^{10}/_{71} < \pi < 3^{10}/_{70}$.

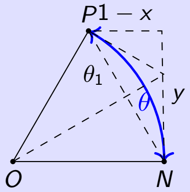
Hipparchus (190 – 120 BCE), Ptolemy (100 – 170 CE)
used chord measure θ_1: θ-equivalent but not additive, for trig tables.

Aryabhatiya (476 – 550 CE)
presented first known additive trigonometric tables.

Euclidean Angles

Angles emanating from O
marked by points N, P
on unit circle centered at O.
Well-defined when N and P not antipodal.

Take $O = (0,0)$, $N = (1,0)$, and
$P = (x, y)$.



Archimedes' angle measure θ:
length of arc along unit circle subtended by angle θ

Treatises on :
Sphere & Cylinder & Measurement of circle

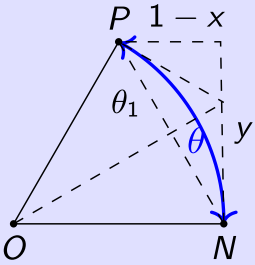
Both computations and *axioms* for:

- Length
- Area
- Volume
- Circle
- Sphere
- Ball

Notions of, A Priori Properties for:

- Convex curve
- Archimedes Postulate
- Additivity of Angle Measure
- Bisection Method
- Completeness of $\mathbb{R}$.

Archimedes' Convexity Postulate



Archimedes convexity postulate, states:

If two convex arcs have common endpoints along a segment and one arc lies below the other then lower has smaller length than upper

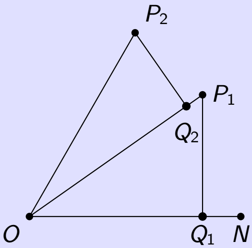
In Angle Figure, applying Archimedes's postulate twice implies

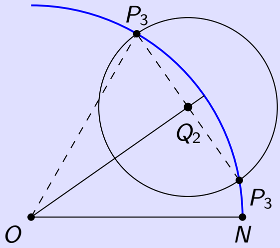
$$y < \theta < 1 - x + y, \qquad (x, y \text{ positive })$$

Angle Stacking

Adding and scaling points in plane:

$$P_1 = (x_1, y_1),\ P_2 = (x_2, y_2); \qquad P_1 + P_2 = (x_1 + x_2, y_1 + y_2); \qquad tP = (tx, ty).$$





Draw circle w center $Q_2 = x_2 P_1$, radius $|y_2|$. For x_2, y_2 nonzero, arc meets circle at

$$P_3 = (x_1x_2 - y_1y_2, x_2y_1 + x_1y_2) \quad \& \quad P_3 = (x_1x_2 + y_1y_2, x_2y_1 - x_1y_2)$$

Thus define $P_1 \times P_2$ and P_1/P_2. Verifiably *commutative* and *associative*.

Multiplicative Properties of Angle Stacking

Conjugate: For $P = (x, y)$, $\bar{P} \equiv (x, -y)$. $N \equiv (0, 1)$; then usual mult rules apply:

$$PN = NP = P; \quad P\bar{P} = N, \; P_1/P_2 = P_1\bar{P}_2.$$

Define *square root* of P, on *punctured circle* $P \neq -N$:

$$\sqrt{P} = \sqrt{(x, y)} \equiv \left(\frac{1 + x}{\sqrt{2 + 2x}}, \frac{y}{\sqrt{2 + 2x}} \right).$$

Verify P and $\sqrt{P}$ have same sign of imaginary part, and:

$$\sqrt{P^2} = P; \quad \left(\sqrt{P}\right)^2 = P; \quad \sqrt{P_1 P_2} = \sqrt{P_1}\sqrt{P_2}\,.$$

Inscribed, Circumscribed Chords

Straight out of Archimedes:

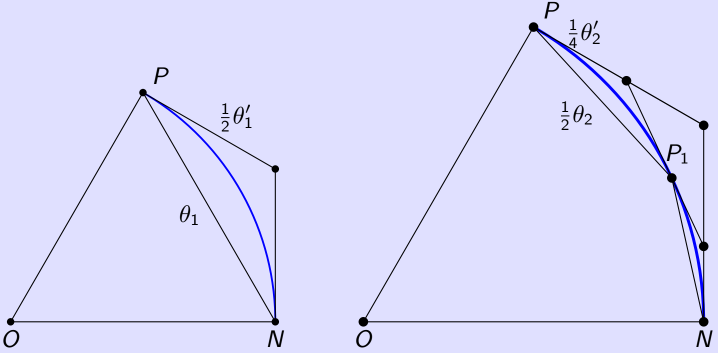


Figure: Inscribed and circumscribed chords.

N, P_1, P 3 pts on unit circle. Subscripted angle symbols are *chord lengths*.

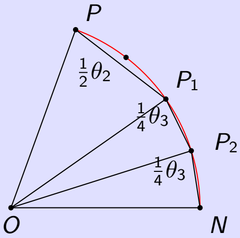
θ_1, θ_1' are inscribed and circumscribed chord-length sums on left;
θ_1, θ_2, θ_2', θ_1' are inscribed and circumscribed chord-length sums on right.
Archimedes' Convexity Postulate applied to 4 polygonal convex curves:

$$\theta_1 < \theta_2 < \theta_2' < \theta_1'.$$

Archimedes Bisection

Incrementally add pts to arc. Iinscribed, circumscribed poly approx of arc; lengths are arbitrarily close to each other:

$$\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_3' < \theta_2' < \theta_1'.$$



Theorem (Uniqueness)

There is at most one real-valued additive angle measure θ on the punctured unit circle satisfying Archimedes' Convexity Postulate.

To prove it, take two candidate angle measures $\phi(), \psi()$, apply them to an n-times bisected arc, and use inscribed and circumscribed polygonal lengths and point P_n to show that $|\phi(P) - \psi(P)|$ differ by a constant times 2^{-n}.

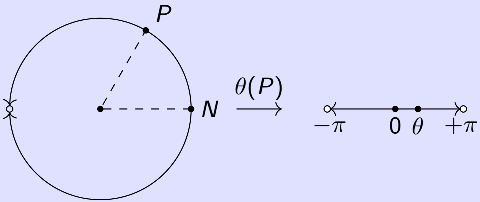
Existence, Continuity, surjectivity

Theorem (Existence)

There is a real-valued additive angle measure θ on the punctured unit circle satisfying Archimedes' Convexity Postulate.

The proof uses the completeness of the reals $\mathbb{R}$ to show that the inscribed and circumscribed approximations converge, first for angles in the first quadrant, and then by extension to general angles. Additivity is proved for approximations and then in general, by passing to the limit.

One then shows that the unique additive angle measure $\theta()$ on the punctured disk is continuous and injective on the right half of the unit circle, and by monotonicity properties, on the entire punctured circle as well.



The Fundamental Theorem of Trigonometry

Theorem (Fundamental theorem of trigonometry)

There is a surjective continuous map $P(\theta)$ of the real line to the unit circle, unique up to rescaling, satisfying

$$P(\theta_1)P(\theta_2) = P(\theta_1 + \theta_2)\,.$$

It suffices to assume non-constancy of $P(\theta)$. Additivity implies surjectivity then. Initially function defined on $(-\pi, +\pi)$. Extend to all real using half-angle square root relation:

$$P(\theta) = P(\theta/2)^2$$

The addition rule holds for the extension.

Continuity is proved using *invariance of domain*.

Say that function $P()$ has α as a *period* if $P(\alpha) = N$. We show:

- $P()$ has a positive period.
- $P()$ has a least, positive period.
- Rescaling $P()$ by its period yield a unique function.