Orthonormal bases and Parseval frames generated by Cuntz algebras and row co-isometries

Dorin Dutkay

Some well known or less known orthonormal bases

Idea of the talk. Various orthonormal bases such as Fourier bases, Walsh bases, Fourier bases on fractals can be constructed by iterating certain isometries, representations of the Cuntz algebra.

• Fourier bases in $L^2[0,1]$: $\{e^{2\pi inx} : n \in \mathbb{Z}\}$.

• Walsh bases in $L^2[0,1]$: orthonormal bases of functions which are ± 1 on dyadic intervals.

• Fourier bases on Cantor measures. The Jorgensen-Pedersen Cantor-4 set. Divide $[0,1]$ into 4 pieces, keep the first and the third. Repeat. Consider μ_4 the Hausdorff measure on the Cantor-4 set. Then

$$\left\{e^{2\pi i\lambda x} : \lambda = l_0 + 4l_1 + \cdots + 4^n l_n, l_i \in \{0,1\}\right\},$$

is an orthonormal basis in $L^2(\mu_4)$.

Fourier series on singular measures

• The Middle Third Cantor set does not work. Question by Strichartz: are there exponential frames for this?
• Generalizations by: Strichartz, Laba-Wang, D-Jorgensen and others.
• Picioroaga-Weber: Parseval frames for singular measures.
• D-Ranasinghe: Walsh Parseval frames.

Goal: Develop a general procedure for all these orthonormal bases and Parseval frames.

The Cuntz algebra

Definition

Let $N \geq 2$. The Cuntz algebra $\mathcal{O}_N$ is the C^*-algebra generated by N isometries $(S_i)_{i=0}^{N-1}$ satisfying the Cuntz relations

$$S_i^* S_j = \delta_{ij}(i, j = 0, \dots, N-1), \quad \sum_{i=0}^{N-1} S_i S_i^* = 1.$$

They are very good for generating orthonormal sets:
If α and β are two words with digits in $\{0, \dots, N-1\}$, and if they differ at some point $\alpha_k \neq \beta_k$ then $S_\alpha H \perp S_\beta H$, where $S_\alpha = S_{\alpha_1} \dots S_{\alpha_n}$.

A general setting for defining Cuntz isometries

Definition

Let X be a compact metric space and μ a Borel probability measure on X. Let $r : X \to X$ be an N-to-1 onto Borel measurable map, i.e. $|r^{-1}(z)| = N$ for μ.a.e. $z \in X$, where $|\cdot|$ indicates cardinality. We say that μ is strongly invariant (for r) if for every continuous function f on X the following invariance equation is satisfied:

$$\int f d\mu = \frac{1}{N} \int \sum_{r(w)=z} f(w) d\mu(z)$$

A general setting for defining Cuntz isometries

Definition

Let X be a compact metric space and μ a Borel probability measure on X. Let $r : X \to X$ be an N-to-1 onto Borel measurable map, i.e. $|r^{-1}(z)| = N$ for μ.a.e. $z \in X$, where $|\cdot|$ indicates cardinality. We say that μ is strongly invariant (for r) if for every continuous function f on X the following invariance equation is satisfied:

$$\int f d\mu = \frac{1}{N} \int \sum_{r(w)=z} f(w) d\mu(z)$$

Example

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let $r(z) = z^N$, $z \in \mathbb{T}$. Let μ be the Haar measure on $\mathbb{T}$. Then μ is strongly invariant. An equivalent system can be realized on$[0,1]$ with $r(x) = Nx \mod 1$, $x \in [0,1]$ with the Lebesgue measure dx on $[0,1]$. We can identify the unit circle $\mathbb{T}$ with the unit interval $[0,1]$ by $z = e^{2\pi i x}$.

A general setting

Example

We consider affine iterated function systems with no overlap. Let R be a $d \times d$ expansive real matrix, i.e., all the eigenvalues of R have absolute value strictly greater than 1.Let $B \subset \mathbb{R}^d$ a finite set such that $N = |B|$. Define the affine iterated function system

$$\tau_b(x) = R^{-1}(x + b) \quad (x \in \mathbb{R}^d,\ b \in B)$$

By Hutchinson there exists a unique compact subset X_B of $\mathbb{R}^d$ which satisfies the invariance equation

$$X_B = \cup_{b \in B} \tau_b(X_B)$$

X_B is called the attractor of the iterated function system $(\tau_b)_{b \in B}$. Moreover X_B is given by

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k \, : \, b_k \in B \text{ for all } k \geq 1 \right\}$$

Also, there is a unique probability measure μ_B on $\mathbb{R}^d$ satisfying the invariance equation

$$\int f d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b d\mu_B$$

for all continuous compactly supported functions f on $\mathbb{R}$. We call μ_B the invariant measure for the IFS $(\tau_b)_{b \in B}$. The measure μ_B is supported on the attractor X_B. We say that the IFS has no overlap if $\mu_B(\tau_b(X_B) \cap \tau_b'(X_B)) = \emptyset$ for all $b \neq b'$ in B.
Assume that the IFS $(\tau_b)_{b \in B}$ has no overlap. Define the map $r : X_B \to X_B$

$$r(x) = \tau_b^{-1}(x), \text{ if } x \in \tau_b(X_B)$$

Then r is an N-to-1 onto map and μ_B is strongly invariant for r. Note that $r^{-1}(x) = \{\tau_b(x) : b \in B\}$ for μ_B.a.e. $x \in X_B$.

A general setting

Definition

Let (X, r) and the measure μ be as above. A *QMF basis* is a set of N functions $m_0, m_1, \dots, m_{N-1}$ in $L^\infty(X)$ such that

$$\frac{1}{N}\sum_{r(w)=z} m_i(w)\overline{m_j}(w) = \delta_{ij}, \quad (i, j \in \{0, \dots, N-1\}, z \in X)$$

Proposition

Let $(m_i)_{i=0}^{N-1}$ be a QMF basis. Define the operators on $L^2(X, \mu)$

$$S_i(f) = m_i f \circ r, \quad i = 0, \dots, N-1$$

Then the operators S_i are isometries and they form a representation of the Cuntz algebra $\mathcal{O}_N$, i.e.

$$S_i^* S_j = \delta_{ij}, \quad i, j = 0, \dots, N-1, \qquad \sum_{i=0}^{N-1} S_i S_i^* = I$$

Examples

Example

(Wavelets) Take $X = [0, 1]$ $r(x) = 2x \mod 1$, μ the Lebesgue measure. Take m_0, m_1 to be the low-pass, high pass filters in multiresolution theory.

Examples

Example

(Wavelets) Take $X = [0,1]$ $r(x) = 2x \bmod 1$, μ the Lebesgue measure. Take m_0, m_1 to be the low-pass, high pass filters in multiresolution theory.

Example

(Fourier series) Take $X = [0,1]$ $r(x) = 2x \bmod 1$, μ the Lebesgue measure. Take $m_0 = 1$, $m_1 = e^{2\pi i x}$.

Examples

Example

(Wavelets) Take $X = [0,1]$ $r(x) = 2x \bmod 1$, μ the Lebesgue measure. Take m_0, m_1 to be the low-pass, high pass filters in multiresolution theory.

Example

(Fourier series) Take $X = [0,1]$ $r(x) = 2x \bmod 1$, μ the Lebesgue measure. Take $m_0 = 1$, $m_1 = e^{2\pi ix}$.

Example

(Walsh series) Take $X = [0,1]$ $r(x) = 2x \bmod 1$, μ the Lebesgue measure. Take $m_0 = 1$, $m_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$.

Example

(Fourier series on fractals) Let R be a $d \times d$ expansive integer matrix, let $B \subset \mathbb{Z}^d$, $\#B = N$, $0 \in B$. Consider $X = X_B$ and $r(x) = Rx (\bmod X_B)$ and the invariant measure μ_B. Assume that there exists a set L in $\mathbb{Z}^d$, with $\#L = N$, $0 \in L$ such that the matrix

$$\frac{1}{\sqrt{N}}(e^{2\pi i R^{-1} b \cdot l})_{l \in L, b \in B}$$

is unitary. (We call (R, B, L) a Hadamard system). Define $m_l(x) = e^{2\pi i l x}$, $l \in L, x \in X_B$. Then $(m_l)_{l \in L}$ forms a QMF basis.

Basic idea

Assume we have some Cuntz isometries $(S_i)_{i=0}^{N-1}$ on a Hilbert space $\mathcal{H}$. In addition assume that $S_0 1 = 1$. Consider the family of vectors

$$\mathcal{E} = \{S_{j_1} S_{j_2} \dots S_{j_n} 1 : j_1, \dots, j_n \in \{0, \dots, N-1\}, n \geq 0\}.$$

To avoid repetitions assume $j_n \neq 0$ and include the empty word, so that 1 appears in the family.

Basic idea

Assume we have some Cuntz isometries $(S_i)_{i=0}^{N-1}$ on a Hilbert space $\mathcal{H}$. In addition assume that $S_0 1 = 1$. Consider the family of vectors

$$\mathcal{E} = \{S_{j_1} S_{j_2} \dots S_{j_n} 1 : j_1, \dots, j_n \in \{0, \dots, N-1\}, n \geq 0\}.$$

To avoid repetitions assume $j_n \neq 0$ and include the empty word, so that 1 appears in the family. The family is always orthogonal, if $j_1 \dots j_n$ and $l_1 \dots l_m$ are two words and $j_k \neq l_k$ for some k then take the first such k, $S_{j_k}\mathcal{H} \perp S_{l_k}\mathcal{H}$ and since the maps are isometries we get that $S_{j_1} \dots S_{j_n} 1 \perp S_{l_1} \dots S_{l_m} 1$. A similar thing can be done when one word is a prefix of the other (complete with 0's at the end). Thus the family is orthonormal.

Basic idea

Completeness is a much more delicate issue. But here is the idea. We consider the norm of projections onto the span, for some "nice" functions e_t which span the entire space (these will usually be exponentials). We have $\mathcal{E} = \cup_{l=0}^{N-1} S_l \mathcal{E}$.

$$h(t) := \|Pe_t\|^2 = \sum_{e\in\mathcal{E}} |\langle e_t\ ,\ e\rangle|^2 = \sum_{l=0}^{N-1}\sum_{e\in\mathcal{E}} |\langle e_t\ ,\ S_l e\rangle|^2 = \sum_{l=0}^{N-1}\sum_{e\in\mathcal{E}} |\left\langle S_l^* e_t\ ,\ e\right\rangle|^2$$

$$= (\text{ some assumption }) = \sum_{l=0}^{N-1}\sum_{e\in\mathcal{E}} |\nu_l(t)|^2 |\left\langle e_{g_l(t)}\ ,\ e\right\rangle|^2 = \sum_{l=0}^{N-1} |\nu_l(t)|^2 h(g_l(t)).$$

Thus h is a fixed point of a Ruelle/transfer operator. If we can show that the only fixed points of this operator are the constant functions then we get $\|Pe_t\| = 1$ which means that the functions e_t are in the span and therefore the family $\mathcal{E}$ is complete.

A general theorem

Theorem

Let $\mathcal{H}$ be a Hilbert space and $(S_i)_{i=0}^{N-1}$ be a representation of the Cuntz algebra $\mathcal{O}_N$. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$ and $e : \mathcal{T} \to \mathcal{H}$ a norm continuous function on a topological space $\mathcal{T}$ with the following properties:

1. $\mathcal{E} = \cup_{i=0}^{N-1} S_i \mathcal{E}$.
2. $\overline{span}\{e(t) : t \in \mathcal{T}\} = \mathcal{H}$ *and* $||e(t)|| = 1$, *for all* $t \in \mathcal{T}$.
3. *There exist functions* $\nu_i : \mathcal{T} \to \mathbb{C}$, $g_i : \mathcal{T} \to \mathcal{T}$, $i = 0, \dots, N-1$ *such that*
$$S_i^* e(t) = \nu_i(t) e(g_i(t)), \quad t \in \mathcal{T}.$$
4. *There exist* $c_0 \in \mathcal{T}$ *such that* $e(c_0) \in \overline{span}\mathcal{E}$.
5. *The only function* $h \in \mathcal{C}(\mathcal{T})$ *with* $h \geq 0$, $h(c) = 1$, $\forall\, c \in \{x \in \mathcal{T} : e(x) \in \overline{span}\mathcal{E}\}$, *and*
$$h(t) = \sum_{i=0}^{N-1} |\nu_i(t)|^2 h(g_i(t)), \quad t \in \mathcal{T}$$
are the constant functions.

Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$.

Forier series on fractals

Consider (R, B, L) a Hadamard system in $\mathbb{Z}$. Let μ_B be the invariant measure associated to (R, B), $r(x) = Rx \bmod X_B$. Recall that $(e_l)_{l\in L}$ is a QMF basis. $e_\lambda(x) = e^{2\pi i\lambda\cdot x}$. The operators

$$(S_l f)(x) = e^{2\pi i l x} f(Rx \bmod X_B), \quad (l \in L, f \in L^2(\mu_B)),$$

are Cuntz isometries. Note that

$$S_l e_k = e_{l+Rk}, \quad (l \in L, k \in \mathbb{Z}).$$

Definition

We say that $c \in \mathbb{R}$ is an *extreme cycle point* for (B, L) if there exists $l_0, l_1, \dots, l_{p-1}$ in L such that, if $c_0 = c$, $c_1 = \frac{c_0+l_0}{R}$, $c_2 = \frac{c_1+l_1}{R} \dots c_{p-1} = \frac{c_{p-2}+l_{p-2}}{R}$ then $\frac{c_{p-1}+l_{p-1}}{R} = c_0$, and $|m_B(c_i)| = 1$ for $i = 0, \dots, p-1$ where

$$m_B(x) = \frac{1}{N}\sum_{b\in B} e^{2\pi i b x} \quad x \in \mathbb{R}.$$

$\{0\}$ is called the trivial cycle.

Theorem

Suppose the only extreme cycle is the trivial one. Then the set

$$\{e_\lambda : \lambda = \sum_{k=0}^{n} R^k l_k, l_k \in L, n \in \mathbb{N}\}$$

is an orthonormal basis for $L^2(\mu_B)$.

Forier series on fractals

If there are some non-trivial cycles, take an extreme cycle c point and apply the Cuntz isometries to e_{-c}. We obtain the following result.

Theorem

Let Λ be the smallest set which contains $-c$ for all extreme cycle points c and which has the invariance property $R\Lambda + L \subset \Lambda$. Then

$$\{e_\lambda : \lambda \in \Lambda\}$$

is an orthonormal basis for $L^2(\mu_B)$.

If there are some non-trivial cycles, take an extreme cycle c point and apply the Cuntz isometries to e_{-c}. We obtain the following result.

Theorem

Let Λ be the smallest set which contains $-c$ for all extreme cycle points c and which has the invariance property $R\Lambda + L \subset \Lambda$. Then

$$\{e_\lambda : \lambda \in \Lambda\}$$

is an orthonormal basis for $L^2(\mu_B)$.

Application: take $R = 2$, $B = \{0, 1\}$ then μ_B is the Lebesgue measure on $[0, 1]$. Take $L = \{0, 1\}$. Since $m_B(x) = (1 + e^{2\pi ix})/2$, we have two extreme cycles $\{0\}$, and $\{1\}$. Applying the theorem, we get the classical Fourier series.

Walsh series on $[0, 1]$

Let $X = [0, 1]$ with the Lebesgue measure, $r(x) = 2x \bmod 1$. The functions $m_0 = 1$ and $m_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ form a QMF basis, so the operators

$$S_0 f(x) = f(2x \bmod 1), \quad S_1 f(x) = m_1(x) f(2x \bmod 1),$$

are Cuntz isometries on $L^2[0, 1]$.

Theorem

The family

$$\{S_{j_1} \dots S_{j_n} 1 : j_1, \dots, j_n \in \{0, 1\}, n \geq 0\}$$

is an orthonormal basis for $L^2[0, 1]$, namely the classical Walsh basis.

Generalized Walsh series on $[0, 1]$

Let $X = [0, 1]$, with the Lebesgue measure $r(x) = Nx \mod 1$. Let A be an $N \times N$ unitary matrix with first row constant $1/\sqrt{N}$. Define the functions

$$m_j = \sqrt{N} \sum_{k=1}^{N} a_{jk} \chi_{[k/n,(k+1)/n)}.$$

Then $(m_j)_{j=1}^{N}$ is a QMF basis so the operators

$$S_j f(x) = m_j(x) f(Nx \mod 1), (j = 1, \dots, N)$$

are Cuntz isometries on $L^2[0, 1]$.

Theorem

The family

$$\{S_{j_1} \dots S_{j_n} 1 : j_1, \dots, j_n \in \{1, 2, \dots, N-1\}, n \geq 0\}$$

is an orthonormal basis for $L^2[0, 1]$.

Parseval frames and Row co-isometries

Definition

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space K, is a Parseval frame if

$$\sum_{i\in I} |\langle v\,,\,f_i\rangle|^2 = \|v\|^2, \quad (v \in H).$$

Theorem

Every Parseval frame is a projection of an orthonormal basis.

Parseval frames and Row co-isometries

Theorem (Popescu, Bratteli et al.)

Let K be a Hilbert space, and let $V_0, \dots, V_{N-1}$ be a row co-isometry, i.e.,

$$\sum_{i=0}^{N-1} V_i V_i^* = I_K$$

Then K can be embedded into a larger Hilbert space H carrying a representation $S_0, \dots, S_{N-1}$ of the Cuntz algebra $\mathcal{O}_N$ such that K is cyclic for the representation, and if $P_K : H \to K$ is the projection onto K we have

$$S_i^*(K) \subset K, \text{ and } V_i^* P_K = S_i^* P_K, V_i = P_K S_{i_{|K}}.$$

Start with a row co-isometry $(V_i)_{i=0}^{N-1}$. Dilate it to a Cuntz representation $(S_i)_{i=0}^{N-1}$. Obtain an orthonormal basis using the S_i's. Project back to obtain a Parseval frame generated by the V_i's.

Assume that there exists a finite set $L \subset \mathbb{Z}$ with $0 \in L$, $|L| =: M$ and complex numbers $(\alpha_l)_{l\in L}$ such that the following properties are satisfied:

1. $\alpha_0 = 1$.
2. The matrix
$$T := \frac{1}{\sqrt{N}} \left(e^{2\pi i R^{-1} l \cdot b} \alpha_l \right)_{l \in L, b \in B} \tag{0.1}$$
is an isometry, i.e., $T^* T = I_N$, i.e., its columns are orthonormal, which means that
$$\frac{1}{N} \sum_{l \in L} |\alpha_l|^2 e^{2\pi i R^{-1} l \cdot (b - b')} = \delta_{b,b'}, \quad (b, b' \in B). \tag{0.2}$$

Construct the components of the row co-isometry

$$V_l f(x) = \alpha_i e^{2\pi i l x} f(r(x)), \quad (x \in X_B, l \in L, f \in L^2(\mu_B)).$$

Parseval frames on fractals

Definition

Let

$$m_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b x}, \quad (x \in \mathbb{R}). \tag{0.3}$$

Aa set $\mathcal{M} \subset \mathbb{R}$ is called *invariant* if for any point $t \in \mathcal{M}$, and any $l \in L$, if $\alpha_l m_B(R^{-1}(t-l)) \neq 0$, then $g_l(t) := R^{-1}(t-l) \in \mathcal{M}$. $\mathcal{M}$ is said to be non-trivial if $\mathcal{M} \neq \{0\}$. We call a finite *minimal invariant* set a *min-set*.

Note that

$$\sum_{l \in L} |\alpha_l|^2 \, |m_B(g_l(t))|^2 = 1 \quad (t \in \mathbb{R}^d), \tag{0.4}$$

and therefore, we can interpret the number $|\alpha_l|^2 \, |m_B(g_l(t))|^2$ as the probability of transition from t to $g_l(t)$, and if this number is not zero then we say that this *transition is possible in one step (with digit l)*, and we write $t \to g_l(t)$ or $t \xrightarrow{l} g_l(t)$. We say that the *transition is possible* from a point t to a point t' if there exist $t_0 = t$, $t_1, \dots, t_n = t'$ such that $t = t_0 \to t_1 \to \cdots \to t_n = t'$. The *trajectory* of a point t is the set of all points t' (including the point t) such that the transition is possible from t to t'.

A *cycle* is a finite set $\{t_0, \dots, t_{p-1}\}$ such that there exist $l_0, \dots, l_{p-1}$ in L such that $g_{l_0}(t_0) = t_1, \dots, g_{l_{p-1}}(t_{p-1}) = t_p := t_0$. Points in a cycle are called *cycle points*.

A cycle $\{t_0, \dots, t_{p-1}\}$ is called *extreme* if $|m_B(t_i)| = 1$ for all i; by the triangle inequality, since $0 \in B$, this is equivalent to $t_i \cdot b \in \mathbb{Z}$ for all $b \in B$.

Let c be an extreme cycle point in some finite minimal invariant set. A word $l_0 \dots l_{p-1}$ in L is called a *cycle word* for c if $g_{l_{p-1}} \dots g_{l_0}(c) = c$ and $g_{l_k} \dots g_{l_0}(c) \neq c$ for $0 \leq k < p-1$, and the transitions $c \to g_{l_0}(c) \to g_{l_1} g_{l_0}(c) \to \cdots \to g_{l_{p-2}} \dots g_{l_0}(c) \to g_{l_{p-1}} \dots g_{l_0}(c) = c$ are possible.

Parseval frames on fractals

For every finite minimal invariant set $\mathcal{M}$, pick a point $c(\mathcal{M})$ in $\mathcal{M}$ and define $\Omega(c(\mathcal{M}))$ to be the set of finite words with digits in L that do not end in a cycle word for $c(\mathcal{M})$, i.e., they are not of the form $\omega\omega_0$ where ω_0 is a cycle word for c and ω is an arbitrary word with digits in L.

Theorem

Suppose (R, B, L) *and* $(\alpha_l)_{l\in L}$ *satisfy the Assumptions. Then the set*

$$\left\{ \left(\prod_{j=0}^{n} \alpha_{l_j}\right) e_{l_0+Rl_1+\cdots+R^k l_k+R^{k+1}c(\mathcal{M})} : l_0 \dots l_n \in \Omega(c(\mathcal{M})), \mathcal{M} \text{ is a min-set} \right\}$$

is a Parseval frame for $L^2(\mu(R, B))$.