Harmonic and Spectral Analysis

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Problems and Remarks

Problem no. 1

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Proposition. *Let Ω be a locally compact, non-compact Hausdorff space and denote $M(\Omega)$ the space of all Radon measures on Ω. Then following statements are equivalent.*

i) *$T^*_{\alpha,w}$ is topologically hyper-transitive on $M(\Omega)$.*

ii) *For every compact subset K of Ω and any two measures μ, v in $M(\Omega)$ with $|v|(K^c) = |\mu|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{A_k\}$, $\{B_k\}$ of compact subsets of K such that $\alpha^{n_k}(K) \cap K = \emptyset$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in K\cap A_k^c} (\prod_{j=0}^{n_k-1}(w \circ \alpha^j)(t)) = \lim_{k\to\infty} \sup_{t\in K\cap B_k^c} (\prod_{j=1}^{n_k}(w \circ \alpha^{-j})^{-1}(t)) = 0.$$

Corollary. *We have that ii) $\Rightarrow$ i)*

i) *$T^*_{\alpha,w}$is topologically hyper-transitive on $M(\Omega)$.*

ii) *For every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{t\in K}(\prod_{j=0}^{n_k-1}(w \circ \alpha^j)(t)) = \lim_{k\to\infty} \sup_{t\in K}(\prod_{j=1}^{n_k}(w \circ \alpha^{-j})^{-1}(t)) = 0.$$

Problem 1. *Does there exist an example where the equivalent conditions of the part ii) in the previous proposition are satisfied, whereas the sufficient conditions of the part ii) in this corollary are not satisfied ?*

Problem no. 2

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Denoting $\mathbf{S}-\mathbf{T}(G)$ (resp. $\mathbf{T}-\mathbf{S}(G)$), if the $Spectral \Rightarrow Tile$ (resp. $Tile \Rightarrow Spectral$) direction of Fuglede's conjecture holds in G. We have the following implication:

$$\mathbf{T}-\mathbf{S}(\mathbb{R}^d) \Longrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}^d) \Longrightarrow \mathbf{T}-\mathbf{S}(G_d),$$

$$\mathbf{S}-\mathbf{T}(\mathbb{R}^d) \Longrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}^d) \Longrightarrow \mathbf{S}-\mathbf{T}(G_d),$$

where G_d can be any Abelian group of d generators.

For $\mathbf{T}-\mathbf{S}$ direction we can say more in the one dimensional case. The following was presented by Dutkay and Lai.

$$\mathbf{T}-\mathbf{S}(\mathbb{R}) \Longleftrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}) \Longleftrightarrow \mathbf{T}-\mathbf{S}(\mathbb{Z}_{\mathbb{N}}).$$

Problem 2. *Is it true that*

$$\mathbf{S}-\mathbf{T}(\mathbb{R}) \Longleftrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}) \Longleftrightarrow \mathbf{S}-\mathbf{T}(\mathbb{Z}_{\mathbb{N}})?$$

The following questions came up working with Gábor Somlai. It would be nice to know the following.

Problem 3. *Given a spectral pair (S,Λ), where $S \subset \mathbb{Z}_N$ for some N. Suppose that Λ is a tile. Is it true that S is also a tile?*

More general question:

Problem 4. *Given a spectral pair (S,Λ), where $S \subset \mathbb{Z}_N$ for some N. Suppose that $|S| \mid N$. Is it true that S is a tile?*

It is known that every tile is spectral in $\mathbb{Z}_p^d$ if $d \le 3$ and p is odd. But it is not even known for $\mathbb{Z}_p^4$. The main question is the following

Problem 5. *Let S be a tile in $\mathbb{Z}_p^4$ and $|S| = p^2$. Is it true that S is spectral?*

Conjecture of Ruxi Shi:

Conjecture. *The tiling-spectral direction holds for $\mathbb{Z}_{p^{l_1}} \times \cdots \times \mathbb{Z}_{p^{l_i}} \times \cdots \times \mathbb{Z}_{p^{l_n}}$, where $n, l_i \in \mathbb{N}$ for $1 \le i \le n$.*

A special case:

Problem 6. *Is it true that every tile is spectral in $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$?*

It is interesting even in case $1 < n, m \le 3$.

Problem no. 3

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For $f \in \mathbb{L}^1(\mathbb{R})$, $f \geq 0$, we define its triple correlation function

$$N_f(x_1, x_2) = \int f(t) f(t + x_1) f(t + x_2) \, \mathrm{d}t.$$

Does N_f determine f up to translations?

Working on the Fourier side it is proved [1] that the answer is no if no conditions are imposed on f. However true when $\hat{f}$ does not vanish on a set of positive measure. Special cases of this are when we know that f is of compact support or $\int f e^{M|x|} < \infty$ for some $M > 0$.

It is also shown in [1] that if $N_f = N_{\mathbf{1}_E}$ and $E \subset \mathbb{R}$ is of finite measure then f is also an indicator function.

Problem 7. *Prove that if $E, F \subset \mathbb{R}$ have finite measure and $N_{\mathbf{1}_E} = N_{\mathbf{1}_F}$, then E is a translate of F.*

Things are not simple on the Fourier side for this case. It is a result of Kargaev [2, 3] that there exist sets of finite measure on the real line whose Fourier Transform (of their indicator function) vanishes on a whole interval.

References

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[3] P. P. Kargaev and A. L. Volberg. Three results concerning the support of functions and their Fourier transforms. *Indiana Univ. Math. J*, 41(4):1143–1164, 1992.

Problem no. 4

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Let f be a measurable function on $\mathbb{R}$ and let $\Lambda \subset \mathbb{R}$ be a countable set. If we have

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = 1 \quad \text{a.e.} \tag{1}$$

and the series in (1) converges absolutely a.e., then we say that $f + \Lambda$ *is a tiling*.

If $f = \mathbf{1}_\Omega$ is the indicator function of a set Ω, then $f + \Lambda$ is a tiling if and only if the translated copies $\Omega + \lambda$, $\lambda \in \Lambda$, fill the whole space without overlaps up to measure zero. To the contrary, for tilings by a general real or complex-valued function f, the translated copies may have overlapping supports.

We say that a set $\Lambda \subset \mathbb{R}$ has *bounded density* if

$$\sup_{x \in \mathbb{R}} \#(\Lambda \cap [x, x+1)) < +\infty. \tag{2}$$

It is known [KL96, Lemma 2.1] that if the function f is *nonnegative* and $f + \Lambda$ is a tiling, then Λ must have bounded density.

The "periodicity theorem" proved in [LM91] and rediscovered in [KL96] states that if $f + \Lambda$ is a tiling where $\Lambda \subset \mathbb{R}$ has bounded density and $f \in L^1(\mathbb{R})$ has *bounded support*, then the translation set Λ must have a periodic structure, namely, it can be represented as a disjoint union of finitely many arithmetic progressions.

To the contrary, we proved in [KL16] that if f is allowed to have unbounded support, then there exist *non-periodic* tilings $f + \Lambda$ where $\Lambda \subset \mathbb{R}$ has bounded density and $f \in L^1(\mathbb{R})$. Moreover, in [KL21] we established the existence of non-periodic tilings by a function f such that $\operatorname{supp}(f) \subset [0, +\infty)$, i.e. the support is *bounded from below* (while it cannot be bounded from both above and below at the same time, due to the periodicity theorem). In fact, the support of f can be localized inside any set which contains arbitrarily long intervals, so $\operatorname{supp}(f)$ can be very sparse.

The following open problems were stated in [KL21].

Problem 8. *Let $f \in L^1(\mathbb{R})$, and suppose that* $\operatorname{supp}(f)$ *has finite measure. If f tiles by a translation set $\Lambda \subset \mathbb{R}$ of bounded density, does it follow that Λ has a periodic structure?*

The periodicity theorem does not apply in this case, since f is *not* assumed to have bounded support.

Problem 9. *Does there exist a measurable set* $\Omega \subset \mathbb{R}$, $0 < \operatorname{mes}(\Omega) < +\infty$, *whose indicator function* $\mathbf{1}_\Omega$ *can tile with a non-periodic translation set* $\Lambda \subset \mathbb{R}$*?*

Notice that such a set Ω (if it exists) must be unbounded, again due to the periodicity theorem.

Problem 10. *Does there exist a function* $f \in L^1(\mathbb{R})$ *which tiles by some translation set* $\Lambda \subset \mathbb{R}$ *of bounded density, but* f *does not admit any tiling by a translation set that has a periodic structure?*

References

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