Harmonic and Spectral Analysis

International Zoom Conference

May 31 – June 2

2021

Problems and Remarks

Remark no. 1
Aichinger's equation

José María Almira
University of Murcia

This remark concerns the problem if compositions and products of generalized polynomials are generalized polynomials as well.

In the paper [1] the authors introduce the following functional equation, that characterizes generalized polynomials of degree $\leq m$:

$$f(x_1 + \cdots + x_{m+1}) = \sum_{i=1}^{m+1} g_i(x_1, x_2, \cdots, \widehat{x_i}, \cdots, x_{m+1}),$$

where $\widehat{x_i}$ means that the function g_i does not depend on x_i. I refer to this equation as *Aichinger's functional equation*. In fact, the following theorem holds:

Theorem 1. *Let G, F be abelian groups and let $f : G \to F$ be a map. The following statements are equivalent:*

(a) *There are functions $g_i : G^m \to F$, $i = 1, 2, \cdots, m+1$ such that f satisfies*

$$f(x_1 + x_2 + \cdots + x_{m+1}) = \sum_{i=1}^{m+1} g_i(x_1, x_2, \cdots, \widehat{x_i}, \cdots, x_{m+1}), \tag{1}$$

where $\widehat{x_i}$ means that g_i does not depend on x_i.

(b) *f satisties Fréchet's functional equation:*$\Delta_{x_1}\Delta_{x_2}\cdots\Delta_{x_{m+1}} f = 0$.

The authors introduce their functional equation in the paper in order to give a nice proof of the fact that if $f : G \to F$, $g : F \to H$ are generalized polynomials of degrees $\leq m, n$ (where G, F, H are commutative groups), respectively, then $g \circ f$ is also a generalized polynomial of degree at most $m \cdot n$. We set $\underline{k} = \{1, 2, \cdots, k\}$ and, for any set X, we define $\mathcal{P}_{\leq m}(X) = \{S \subseteq X : |S| \leq m\}$. Then the following lemma can be proved by easy calculation.

Lemma 1. *Let $f : G \to H$ be a map and assume that G, H are commutative groups. Then f satisfies Fréchet's functional equation $\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_{m+1}} f = 0$ if and only if for every integer $k \geq 1$ there exist integers α_S, $S \in \mathcal{P}_{\leq m}(\underline{k}) = \{S \subseteq \{1, 2, \cdots, k\} : \#S \leq m\}$, such that*

$$f(x_1 + \cdots + x_k) = \sum_{S \in \mathcal{P}_{\leq m}(\underline{k})} \alpha_S f(\sum_{i \in S} x_i). \tag{2}$$

Now we are in the position to prove the following theorem:

Theorem 2 (Leibman, 1999, Aichinger–Moosbauer, 2021)**.** *Let G, H, E be commutative groups and $f : G \to H$, $g : H \to E$ polynomial functions. Then* $\deg(g \circ f) \leq \deg(f) \cdot \deg(g)$.

Proof. Assume that $\deg(f) = m$, $\deg(g) = n$ and let $x_1, \cdots, x_{nm+1} \in G$. Then Lemma 1 implies that

$$f(x_1 + \cdots + x_{nm+1}) = \sum_{S \in \mathcal{P}_{\leq m}(\underline{nm+1})} \alpha_S f(\sum_{i \in S} x_i),$$

so that

$$g(f(x_1 + \cdots + x_{nm+1})) = g(\sum_{S \in \mathcal{P}_{\leq m}(\underline{nm+1})} \alpha_S f(\sum_{i \in S} x_i))$$

and, if we set $y_S = \alpha_S f(\sum_{i \in S} x_i)$ and we use Lemma 1 with g, we get

$$g(f(x_1 + \cdots + x_{nm+1})) = g(\sum_{S \in \mathcal{P}_{\leq m}(\underline{nm+1})} y_S) =$$

$$\sum_{T \in \mathcal{P}_{\leq n}(\mathcal{P}_{\leq m}(\underline{nm+1}))} \beta_T g(\sum_{S \in T} y_S) = \sum_{T \in \mathcal{P}_{\leq n}(\mathcal{P}_{\leq m}(\underline{nm+1}))} \beta_T g(\sum_{S \in T} \alpha_S f(\sum_{i \in S} x_i)).$$

Now, for each $T \in \mathcal{P}_{\leq n}(\mathcal{P}_{\leq m}(\underline{nm+1}))$, the function $g(\sum_{S \in T} \alpha_S f(\sum_{i \in S} x_i))$ depends only on the variables x_i whose indices i belong to $\bigcup_{S \in T} S$ and each one of these sets S have cardinal $\leq m$ and the cardinal of T is $\leq n$. Hence, the corresponding function depends on at most nm variables x_i with $1 \leq i \leq nm + 1$. Hence, grouping the functions that depend on the same variables, the expression above proves that $(g \circ f)(\sum_{i=1}^{nm+1} x_i))$ satisfies Aichinger's equation and $\deg(g \circ f) \leq nm = \deg(f)\deg(g)$.

□

Concerning the product of polynomial functions defined on a $\mathbb{Q}$-vector space X things are easier since, if $f_i = diag(A^i)$ and $g_j = diag(B^j)$ are monomials defined on X with values in $\mathbb{C}$ (or any other field extension $\mathbb{K}$ of $\mathbb{Q}$), then $f_i(x)g_j(x) = diag(C^{i,j})(x)$, where

$$C^{i,j}(x_1, \cdots, x_i, y_1, \cdots, y_j) = A^i(x_1, \cdots, x_i)B^j(y_1, \cdots, y_j)$$

is $(i + j)$-additive. Hence, if $f, g : X \to \mathbb{K}$ are polynomial functions , $f = f_0 + \cdots + f_n$, $g = g_0 + \cdots + g_m$, then

$$(f \cdot g)(x) = \sum_{k=0}^{n+m} \sum_{i+j=k} f_i(x)g_j(x) = \sum_{k=0}^{n+m} \sum_{i+j=k} diag((C^{i,j})^{sym})(x)$$

is a polynomial function of degree at most $n + m$. Thus we have the following result:

Proposition 1. *Let X, Y, Z be $\mathbb{Q}$-vector spaces and let $\mathbb{K}$ be any field extension of $\mathbb{Q}$. Assume that $f, g : X \to \mathbb{K}$ and $F : X \to Y$, $G : Y \to Z$ are polynomial functions . Then $f \cdot g$ and $G \circ F$ are polynomial functions . Furthermore,*

$$\deg(f \cdot g) \leq \deg(f) + \deg(g)$$

and

$$\deg(G \circ F) \leq \deg G \cdot \deg F.$$

If we use $\Delta_h^{m+1} f = 0$ as the definition of polynomial function of degree $\leq m$, then we can prove, by induction on m, the formula

$$\Delta_h^m (f \cdot g)(x) = \sum_{i=0}^{m} \binom{m}{i} \Delta_h^i f(x) \cdot \Delta_h^{m-i} g(x + ih)$$

and, as a corollary, we get that $\deg(f \cdot g) \leq \deg(f) + \deg(g)$. Indeed, if $\deg(f) = n$, $\deg(g) = m$, then

$$\Delta_h^{n+m+1} (f \cdot g)(x) \quad = \quad \sum_{i=0}^{n+m+1} \binom{n+m+1}{i} \Delta_h^i f(x) \cdot \Delta_h^{n+m+1-i} g(x + ih) = 0$$

since $i \leq n$ implies $m + 1 \leq n + m + 1 - i$, which implies that all summands in the second member of the equality above vanish. The same idea can be used to prove the following results:

Proposition 2. *Assume that $f, g : X \to \mathbb{K}$ are generalized monomials of degrees n, m, respectively. Then their product is a generalized monomial of degree $n + m$, and the same holds with products of the form $f(x)g(y)$ with $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$ generalized monomials.*

Proof. Recall that f is a generalized monomial of degree n if (and only if)

$$\frac{1}{n!} \Delta_h^n f(x) = f(h).$$

Now we have:

$$\Delta_h^{n+m} (f \cdot g)(x) = \sum_{i=0}^{n+m} \binom{n+m}{i} \Delta_h^i f(x) \cdot \Delta_h^{n+m-i} g(x + ih) =$$

$$\sum_{i=0}^{n-1} \binom{n+m}{i} \Delta_h^i f(x) \cdot \Delta_h^{n+m-i} g(x + ih) + \binom{n+m}{n} \Delta_h^n f(x) \cdot \Delta_h^m g(x + nh) +$$

$$\sum_{i=n+1}^{n+m} \binom{n+m}{i} \Delta_h^i f(x) \cdot \Delta_h^{n+m-i} g(x + ih) = \binom{n+m}{n} \Delta_h^n f(x) \cdot \Delta_h^m g(x + nh) =$$

$$\frac{(n+m)!}{n!m!} n! f(h) \cdot m! g(h) = (n+m)!(f \cdot g)(h),$$

which proves that $f \cdot g$ is a generalized monomial of degree $n+m$. For the second claim, it is enough to take into account that $\phi(x, y) = f(x)g(y)$ can be written as $\phi = \phi_1 \cdot \phi_2$ where $\phi_1(x, y) = f(x)$ and $\phi_2(x, y) = g(y)$, and that, under this notation, we have that

$$\Delta_{(h,k)}^s \phi_1(x, y) = \Delta_h^s f(x) \text{ and } \Delta_{(h,k)}^s \phi_2(x, y) = \Delta_h^s g(y)$$

and use Leibniz's formula por $\phi_1 \cdot \phi_2$:

$$\Delta^{n+m}_{(h,k)}(\phi)(x,y) = \sum_{i=0}^{n+m} \binom{n+m}{i} \Delta^{i}_{(h,k)}\phi_1(x,y) \cdot \Delta^{n+m-i}_{(h,k)}\phi_2((x,y)+i(h,k)) =$$

$$\sum_{i=0}^{n+m} \binom{n+m}{i} \Delta^{i}_{h} f(x) \cdot \Delta^{n+m-i}_{k} g(x+ik) = \binom{n+m}{n} \Delta^{n}_{h} f(x) \cdot \Delta^{m}_{k} g(x+nk) =$$

$$\frac{(n+m)!}{n!m!} n! f(h) \cdot m! g(k) = (n+m)! f(h)g(k) = (n+m)!\phi(h,k).$$

□

Proposition 3 (Homogeneization)**.** *Let $\mathbb{K}$ be any field extension of $\mathbb{Q}$ and assume that $f : \mathbb{K}^n \to \mathbb{K}$ is a polynomial function of degree m. Then $g(x,y) = y^m f(\frac{x}{y})$ $(y \neq 0)$ can be extended to a generalized monomial of degree m in $\mathbb{K}^{n+1}$, named homogeneization of f. Moreover, if $f : \mathbb{K}^{n+1} \to \mathbb{K}$ is a generalized monomial of degree m, then $g(x_1, \cdots, x_m) = f(x_1, \cdots, x_m, 1)$ is a polynomial function of degree m whose homogeneization is f.*

Proof. Let $f(x) = A^0 + diag(A^1)(x) + \cdots + diag(A^m)(x)$ be the canonical representation of f. Then Proposition 2 implies that

$$\begin{aligned} g(x,y) &= y^m (A^0 + diag(A^1)(x/y) + \cdots + diag(A^m)(x/y)) \\ &= y^m \cdot A^0 + y^{m-1} \cdot diag(A^1)(x) + \cdots + y \cdot diag(A^{m-1})(x) + diag(A^m)(x) \end{aligned}$$

is a sum of generalized monomials of degree m, so that it is also a generalized monomial of degree m. On the other hand, if $f(x_1, \cdots, x_{n+1})$ is a generalized monomial of degree m, $g(x_1, \cdots, x_n) = f(x_1, \cdots, x_n, 1)$ is obviously a polynomial function of degree $\leq m$ since

$$\Delta^{m+1}_{(h_1,\cdots,h_n)} g(x_1, \cdots, x_n) = \Delta^{m+1}_{(h_1,\cdots,h_n,0)} f(x_1, \cdots, x_n, 1) = 0 \text{ for all } (h_1, \cdots, h_n) \in \mathbb{K}^n$$

Moreover, the homogeneization of g is given by

$$h(x_1, \cdots, x_n, x_{n+1}) = x^m_{n+1} g(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}) = x^m_{n+1} f(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}, 1) =$$

$$f(x_{n+1}\frac{x_1}{x_{n+1}}, \cdots, x_{n+1}\frac{x_n}{x_{n+1}}, x_{n+1}) = f(x_1, \cdots, x_{n+1}).$$

□

Theorem 2 was generalized by Leibman to the non-commutative case -with no direct use of Aichinger's equation, but with similar arguments-. Moreover, Leibman also proved that if we consider a composition of several polynomial functions $f_i : G_{i-1} \to G_i$, $i = 1, 2, ..., k$, and G_k is nilpotent (with no other extra hypothesis on the first groups $G_0, G_1, ..., G_{k-1}$, that may be non-commutative and quite general, indeed), then the composition $f_k \circ f_{k-1} \circ \cdots \circ f_1$ is also a polynomial (with no precise estimation of its degree).

Aichinger and Moosbauer call functional degree to what we usually call degree. I think it is a good choice of name, since – as they already point in their paper – ordinary polynomials defined on finite groups can have a functional degree smaller than their formal degree as a polynomial (e.g. as an element of $Z_p[x]$). They also introduce a concept of partial (functional) degree and prove reasonable properties for these concepts. Moreover, they also use these properties to demonstrate several nice results in algebraic geometry. So, they get a very natural application of the functional equations approach to polynomials.

As far as I know, Aichinger's equation has not been studied by functional equations' community and my impression is that its study – for example, the study of its stability properties – may be of interest. In any case, it is an example of a functional equation that appears naturally in algebra and could have new interesting applications.

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Address:
José M. Almira
Departamento de Ingeniería y Tecnología de Computadores
University of Murcia
Murcia
Spain
E-mail: jmalmira@um.es

Problem no. 1

STEFAN IVKOVIČ

The Mathematical institute of the Serbian Academy of Sciences and Arts

Let X be Banach space and A, B be bounded, linear closed range operators on X. Suppose in addition that AB also has closed range. Consider the product of $\ker A$ and $\ker B$, which is also a Banach space. Does it hold in general that $\ker AB$ is isomorphic to a closed subspace of the product of $\ker A$ and $\ker B$?

If $\ker B$ is complementable, then it is straightforward to show this, but I do not know for the general case.

Address:
Stefan Ivkovič
Institute of Mathematics
Serbian Academy of Sciences and Arts (SANU)
Knez Mihailova 35
11000 Belgrade
Serbia
E-mail: stefan.iv10@outlook.com

Problem no. 2

Stefan Ivkovič
The Mathematical institute of the Serbian Academy of Sciences and Arts

Are there two finitely generated Hilbert C^*-modules M and N such that M is a submodule of N and such that N is isomorphic to a closed submodule of M, but such that M and N are not isomorphic to each other?

If the answer is negative, then as an application one can show that the intersection of the set of upper semi-C^*-Weyl operators and the set of lower semi-C^*-Weyl operators is the set of C^*-Weyl operators (as one has for the classical semi-Weyl operators on Hilbert spaces).

Address:
Stefan Ivkovič
Institute of Mathematics
Serbian Academy of Sciences and Arts (SANU)
Knez Mihailova 35
11000 Belgrade
Serbia
E-mail: stefan.iv10@outlook.com

Problem no. 3
Spectrality of product domains

Nir Lev
Bar-Ilan University

Recall that a bounded, measurable set $\Omega \subset \mathbb{R}^d$ is said to be *spectral* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that the system of exponential functions $\{\exp 2\pi i\langle\lambda, x\rangle\}$, $\lambda \in \Lambda$, is orthogonal and complete in the space $L^2(\Omega)$. Any set Λ with this property is called a *spectrum* for Ω.

It is not difficult to show that if $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ are two spectral sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then their cartesian product $\Omega = A \times B$ is a spectral set in $\mathbb{R}^n \times \mathbb{R}^m$. Indeed, if $U \subset \mathbb{R}^n$ is a spectrum for A, and $V \subset \mathbb{R}^m$ a spectrum for B, then the product set $\Lambda = U \times V$ serves as a spectrum for Ω (see, for example, [3, Theorem 3]).

In [4] the question was posed as to whether the converse statement is also true.

Conjecture 1. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two bounded, measurable sets. Then their product $\Omega = A \times B$ is spectral if and only if A and B are both spectral sets.*

The "only if" part of this conjecture is the non-trivial one. The difficulty lies in that we assume the product set Ω to be spectral, but we do not make any a priori assumption that the spectrum Λ also has a product structure, so it is not obvious which sets U and V may serve as spectra for the factors A and B, respectively. (One can show, see [3, Lemma 2], that if Ω admits a spectrum Λ with a product structure, $\Lambda = U \times V$, then U is a spectrum for A and V is a spectrum for B.)

It was proved in [1] that Conjecture 1 holds in the case where one of the factors, say A, is an interval in $\mathbb{R}$. Kolountzakis [4] established, using a different approach, that the conjecture is true also if the set A is the union of two intervals in $\mathbb{R}$. In [2] the conjecture was proved in the case where the factor A is a convex polygon in the plane $\mathbb{R}^2$.

One reason to expect that Conjecture 1 should be true is that the analogous assertion for translational tiling is known to hold. Indeed, the product set $\Omega = A \times B$ can tile the space $\mathbb{R}^n \times \mathbb{R}^m$ by translations if and only if both A tiles $\mathbb{R}^n$ and B tiles $\mathbb{R}^m$, see [4, Section 1.2]. It is well-known that in many respects, spectral sets "behave like" sets which can tile the space by translations. So the analogy between spectrality and tiling suggests that Conjecture 1 should be true.

The problem can also be posed in the context of finite abelian groups. Namely, let G, H be two finite abelian groups, and let $A \subset G$ and $B \subset H$. If the cartesian product $A \times B$ is a spectral set in the group $G \times H$, must each one of the factors A, B be spectral in its respective space?

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Address:
Nir Lev
Department of Mathematics
Bar-Ilan University
Ramat Gan 52900
Israel
E-mail: levnir@math.biu.ac.il

Problem no. 4
Multi-tiling set and union of fundamental domains

AZITA MAYELI
The Graduate Center and Queensborough of the City University of New York
(jointly with ALEX IOSEVICH)

Let $\Omega \subset \mathbb{R}^d$ be a set with with positive and finite Lebesgue measure. We say Ω is a multi-tiling set at an integer level $\ell \geq 1$, if there is a countable and discrete set $A \subset \mathbb{R}^d$ such that

$$1_\Omega * 1_A(x) = \ell \quad \text{a.e. } x \in \mathbb{R}^d.$$

The question that we are interested to answer to is:

Is it true that any domain of Euclidean space that multi-tiles is a union of suitable deformations of a fundamental domain with respect to some lattice?

Addresses:
Azita Mayeli
The Graduate Center and Queensborough
City University of New York
New York
USA
E-mail: Amayeli@gc.CUNY.edu

Alex Iosevich
Department of Mathematics
University of Rochester
Rochester NY 14627
USA
E-mail: iosevich@math.rochester.edu

Problem no. 5
Fuglede–Gabor problem

AZITA MAYELI
The Graduate Center and Queensborough of the City University of New York

Let $\Omega \subset \mathbb{R}^d$ be a domain of positive and finite measure. Let $S \subset \mathbb{R}^d \times \mathbb{R}^d$ be a countable and discrete subset. The *Gabor system* generated by χ_Ω is defined as

$$\mathcal{G}(\chi_\Omega, S) := \left\{ e^{2\pi i x \cdot a} \chi_\Omega(x - b) : (a, b) \in S \right\}.$$

For any $(a, b) \in S$, the parameter a is considered as translation (shift) parameter in frequency domain and b is translation parameter in time domain. It is trivial that the role of time and frequency shifts change under the Fourier transform: Let $g_{a,b}(x) = e^{2\pi i x \cdot a} \chi_\Omega(x - b)$, then

$$\hat{g}_{a,b}(\xi) = e^{-2\pi i b \cdot \xi} \hat{\chi}_\Omega(\xi - a)$$

Using this simple observation, we introduce a problem in [LM19] that connects the tiling and spectral properties of a domain K to the property of the Gabor system generated by χ_K, in terms of orthogonal basis. The problem was dubbed the Fuglede-Gabor Problem, and it goes as follows:

Problem (Fuglede-Gabor Problem). *Assume that $\Omega \subset \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ are given as above. Let χ_Ω denote the characteristic function of Ω. If $\mathcal{G}(\chi_\Omega, S)$ is an orthogonal basis for $L^2(\mathbb{R}^d)$, then Ω tiles $\mathbb{R}^d$ by translations and $L^2(\Omega)$ has an orthogonal basis of exponentials.*

It is obvious that the problem's inverse is always true. Assume that Ω tiles the space via T translations and that $L^2(\Omega)$ has an orthogonal exponential basis of type $\mathcal{E}(\Lambda) := \{e^{2\pi i x \cdot \lambda} : \lambda \in \Lambda\}$. Take $S = T \times \Lambda$. Then $\mathcal{G}(\chi_\Omega, S)$ is an orthogonal basis for $L^2(\mathbb{R}^d)$. Furthermore, the Fuglede-Gabor problem is always true if S is separable, that is, if it takes the form $S = A \times B$.

The Fuglede Conjecture is known to hold for lattices. The hypothesis states that every domain K of positive and finite measure in the Euclidean space $\mathbb{R}^d$ tiles the space by translations if and only if $L^2(K)$ has an orthogonal exponential basis. Intuitively, the Fuglede-Gabor problem should hold to lattices as well, i.e. when S is a lattice. When S is a full lattice but not in the form of $A \times B$, some strategies must be used to solve the problem. In [2], the authors solved the problem for scenarios when S is a lower or upper triangular lattice with some prior knowledge of the lattice points. The problem remains unsolved in its entirety.

In [1], the problem's analogue in finite settings was studied, and some preliminary results were obtained.

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Address:
Azita Mayeli
The Graduate Center and Queensborough
City University of New York
New York
USA
E-mail: Amayeli@gc.CUNY.edu

Problem no. 6
A structure versus randomness problem in $(\mathbb{Z}/p\mathbb{Z})^3$

Sam Mattheus
Vrije Universiteit Brussel

We are looking for a dichotomy for a set of p^2 points in $(\mathbb{Z}/p\mathbb{Z})^3$ of the following form:

either S has structure **or** its Fourier transform has large support.

By the Fourier transform of a set, we mean the Fourier transform of its indicator function 1_S and hence might be denoted as $\widehat{1_S}$. Large in this setting means at least cp^3 for some $0 < c < 1$.

The exact notion of *structure* has several possibilities. One natural, but very strong notion is that S is the union of p lines. These lines need not be parallel: the union of k lines in a parallel class and $p-k$ lines in a different one (all disjoint) has $|\text{supp}(\widehat{1_S})| = 2p^2 - 1$. We thank Mihalis Kolountzakis for this example. Maybe this notion is too strong and counterexamples might exist. A weaker, and hence possibly easier to prove, version of structure might be that S contains a line.

There exist several analogous results in other instances and the problem itself might be generalized in several directions (different size of S, other dimensions, ...):

- p points in $(\mathbb{Z}/p\mathbb{Z})^2$ is either a line or $|\text{supp}(\widehat{1_S})|$ is at least (roughly) $(1/2)p^2$, which is Rédei's theorem reformulated;
- a similar result can be derived for sets of kp points in $(\mathbb{Z}/p\mathbb{Z})^2$ by the uncertainty principle (Theorem 1) in [1];
- p points in $(\mathbb{Z}/p\mathbb{Z})^3$ lie in a plane or $|\text{supp}(\widehat{1_S})|$ is at least (roughly) $(7/9)p^3$ [2].

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Address:
Sam Mattheus
Department of Mathematics (Vakgroep Wiskunde)
Vrije Universiteit Brussel
1050 Elsene, Brussels
Belgium
E-mail: sam.mattheus@vub.ac.be