Harmonic and Spectral Analysis

International Zoom Conference

June 8–10 2020

Problems and Remarks

Problem no. 2

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Let $\mathcal{A}$ be a unital C^*-algebra and $H_{\mathcal{A}}$ denote the standard module over $\mathcal{A}$, that is $H_{\mathcal{A}} = l_2(\mathcal{A})$. We have the following proposition.

Proposition. *Let $\mathcal{A}$ be a unital C^*-algebra $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is unilateral shift and $S^*e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1))$ or if $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}$, (where in the case when $\mathcal{A} = L^\infty((0,1))$, we set* $\inf|\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}[0,C]) > 0\} = \sup\{K > 0 \mid |\alpha| > K)$ *a.e. on* $(0,1)\}$). *Moreover, $\sigma_p^{\mathcal{A}}(S) = \emptyset$ in both cases.*

Problem. *If $\mathcal{A} = B(H)$, can we give an explicit description of the generalized $\mathcal{A}$-valued spectrum of S as done in the proposition above?*

Consider again the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider bilateral shift operator V w.r.t. this basis i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^*e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition. *Let V be bilateral shift operator. Then the following holds*
1) If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \emptyset\}$
2) If $\mathcal{A} = L^\infty([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon)) > 0\ \forall\epsilon > 0\}$. In both cases $\sigma_p^{\mathcal{A}}(V) = \emptyset$.

Problem. *If $\mathcal{A} = B(H)$, can we give an explicit description of the generalized $\mathcal{A}$-valued spectrum of V in this case?*

Remark. Recenlty Professor Ivkovič provided the solution of the first part of the suggested problem. All the details about this can be found at arXiv:2007.05237. For the sake of completeness, here we also cite his arguments.

Lemma 1. *Let $T \in B(H)$ and suppose that T is invertible. Then the equation $(TI - S)x = y$ has a solution in $H_{\mathcal{A}}$ for all $e_k, k \in \mathbb{N}$ if and only if the sequence $(T^{-1}, T^{-2}, \ldots, T^{-k}, \ldots)$ is in $H_{\mathcal{A}}$.*

Proof. For $k = 1$, if $(TI - S)x = e_1$ we must have $TB_1 = I$ where $x = (B_1, B_2, \ldots)$. Hence $B_1 = T^{-1}$. Next $TB_2 - B_1 = 0$, so $TB_2 = B_1 = T^{-1}$ which gives $B_2 = T^{-2}$. Proceeding inductively, we obtain that $B_k = T^{-k}$ for all k. So the equation $(TI - S)x = e_1$ has a solution in $H_{\mathcal{A}}$ if and only if the sequence $T^{-1}, T^{-2}, \ldots$ belongs to $H_{\mathcal{A}}$.

Now, if $T^{-1}, T^{-2}, \ldots \in H_{\mathcal{A}}$ then the sequence $x^{(k)}$ in $H_{\mathcal{A}}$ given by

$$x_n^{(k)} = \begin{cases} 0 & \text{if } n \in \{1, \ldots, k-1\} \\ T^{-(n-k+1)} & \text{for } n \in \{k, k+1, \ldots\} \end{cases}$$

is the solution of the equation $(TI - S)x = e_k$ for each $k \in \mathbb{N}$. □

For each $T \in B(H)$ let N_T denote the set of all right annihilators of T. For each right invertible T, let B_T denote right inverse of T and W be the set of all right invertible $T \in B(H)$ such that there exists a non zero sequence $\{Y_n\} \subseteq N_T$ with the property that the sequence $\{Y_1, B_T Y_1 + Y_2, B_T^2 Y_1 + B_T Y_2 + Y_3, \ldots\}$ belongs to $l_2(B(H))$.

In addition, let Z be the set of all right invertible T such that there is **no** sequence $\{Y_n\}$ in N_T with the property that the sequence $\{B_T + Y_1, B_T^2 + B_T Y_1 + Y_2, B_T^3 + B_T^2 Y_1 + B_T Y_2 + Y_3, \ldots\}$ belongs to $l_2(B(H))$. Then we have the following proposition.

Proposition 1. *Let $\mathcal{A} = B(H)$ and S be the unilateral shift on $H_{\mathcal{A}}$. Then*

$$\sigma^{\mathcal{A}}(S) = Z \cup W \cup \{T \in B(H) \,|\, T \text{ is not right invertible}\}.$$

Proof. If $(TI - S)x = y$ should have a solution for each e_k, then, by the similar calculations as in the proof of Lemma 1 we must have that T is right invertible and belongs to $B(H) \setminus Z$. Now, suppose that T is right invertible and that $(TS - I)y = 0$ for some $y = \{Y_1, Y_2, \ldots\} \in H_{\mathcal{A}}$. We get coordinatewise the system of equations $TY_1 = 0, TY_2 = Y_1, TY_3 = Y_2, \ldots$. It is easily seen that $(TS - I)y = 0$ has a non trivial solution if and only if T belongs to W. The proposition follows. □

Corollary 1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then*

$$\sigma^{\mathcal{A}}(S) = \mathcal{A} \setminus G(\mathcal{A}) \cup \{\alpha \in G(\mathcal{A}) \,|\, (\alpha^{-1}, \alpha^{-2}, \ldots, \alpha^{-k}, \ldots) \notin H_{\mathcal{A}}\}.$$

Proof. Since $\mathcal{A}$ is commutative, then the set of right invertible elemnts is $G(\mathcal{A})$. Hence we can apply the arguments from the proof of Lemma 1. □

Corollary 2. *Let $\mathcal{A}$ be a unital C^*-algebra. If $1_{\mathcal{A}}$ denotes the unit in $\mathcal{A}$, then $1_{\mathcal{A}} \in \sigma^{\mathcal{A}}(S)$.*

Proof. We have obviously that the sequence $(1_{\mathcal{A}}, 1_{\mathcal{A}}, 1_{\mathcal{A}}, \ldots) = (1_{\mathcal{A}}^{-1}, 1_{\mathcal{A}}^{-2}, 1_{\mathcal{A}}^{-3}, \ldots)$ is not an element of $H_{\mathcal{A}}$. Then apply the arguments from the proof of Lemma 1. □

Example 1. We may also consider a weighted shift S_w on $H_{\mathcal{A}}$ given by $S_w(x)_{j+1} = w_j x_j$ where $w = (w_1, w_2, \ldots)$ is a bounded sequence in $\mathcal{A}$. In this case, if α has a common right annihilator as w_j for some $j \in \mathbb{N}$, then the sequence having this right annihilator in its j-th coordinate and 0 elsewhere belongs to the kernel of $\alpha I - S_w$. Hence $\alpha \in \sigma^{\mathcal{A}}(S_w)$ in this case.

Remark. Notice that Proposition 1 can be generalized to arbitrary unital C^*-algebras.