Tensor products of synthesizable modules and a question on varieties with spectral synthesis

Bettina Wilkens

University of Namibia, Windhoek

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Let $G$ be an abelian group and $\mathcal{C}G$ the space of functions $G \to \mathbb{C}$, endowed with the topology of pointwise convergence. By linear extension, $\mathcal{C}G$ emerges as isomorphic to the (algebraic) dual space of $\mathcal{C}G$ whereas, in turn, $\mathcal{C}G$ is isomorphic to the space of continuous linear functionals $\mathcal{C}G \to \mathbb{C}$ - the latter isomorphism effected by identifying an element $u$ in $\mathcal{C}G$ with the function $f \mapsto f(u)$, where $f \in \mathcal{C}G^*$. The group $G$ acts on $\mathcal{C}G$ via $(xf)(g) = f(xg)$. The closed submodules of $\mathcal{C}G$ with respect to this action are called varieties on $G$. Thanks to the Hahn-Banach theorem, we have:

**Theorem 1**

If $V$ is a variety on $G$, then $V$ is isomorphic to the dual of $\mathcal{C}G/V^\perp$, where $V^\perp$ is the ideal $\{u \in \mathcal{C}G \mid h(u) = 0 \text{ for } h \in V\}$. 
Let $G$ and $H$ be abelian groups. The group algebra $\mathbb{C}[G \times H]$ is isomorphic to the tensor product of $\mathbb{C}G$ with $\mathbb{C}H$. Let $V$ and $W$ be varieties on $G$ and on $H$, respectively. Mapping a tensor $f \otimes g$ to the map $xy \mapsto f(x)g(y)$ ($x \in G$, $y \in H$) gives rise to a submodule of $\mathbb{C}[G \times H]$ that is isomorphic to $V \otimes W$, $G \times H$ acting on $V \otimes W$ via $xy(v \otimes w) = xu \otimes yv$. But is that submodule a variety?
Let $V = \mathbb{C}G$ and $W = \mathbb{C}H$. Let $0 \neq u \in \mathbb{C}G \otimes \mathbb{C}H$. The element $u$ may be written as a sum $\sum_{i=1}^{n} a_i \otimes b_i$ with linearly independent elements $a_1, \ldots, a_n$ of $V$ and nonzero elements $b_1, \ldots, b_n$ of $W$. Let $j \in \{1, \ldots, n\}$. There is $f \in \mathbb{C}G$ satisfying $f(a_j) = 1$ and $f(a_i) = 0$ if $i \neq j$. For $g \in \mathbb{C}H$, $\sum_{i=1}^{n} f(a_i)g(b_i) = f(a_j)g(b_j)$. So if $u \in (V \otimes W)^\perp$, then $g(b_j) = 0$ for all $g \in W$, which means that $b_j = 0$, a contradiction. It follows that the annihilator ideal $V \otimes W$ is the zero ideal. However, if both $G$ and $H$ are infinite, then the dual of $\mathbb{C}G \otimes \mathbb{C}H$ contains elements that are not linear combinations of tensors $f \otimes h$ with $f \in V$ and $h \in W$. 
The closure $V \otimes W$ is a variety for the direct product $G \times H$. We outline a proof of the following

**Theorem 2**

If $V$ and $W$ are synthesizable, then so is the closure of $V \otimes W$. 
A $\mathbb{C}$-algebra $A$ is residually finite-dimensional if every nonzero element of $A$ is nonzero in a finite quotient of $A$. A direct corollary of theorem 1 is that

A variety $V$ on $G$ is synthesizable (i.e. the closure of its finite-dimensional subvarieties) if and only if the ring $\mathbb{C}G/V_{\perp}$ is residually finite dimensional.

**Lemma 1**

Let $R$ be a $\mathbb{C}$-algebra. If $R$ is residually finite-dimensional, then finitely many linearly independent elements of $R$ are linearly independent over some finite-codimension ideal of $R$. 

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Proof:
Let \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \) linearly independent elements of \( R \). We proceed via induction on \( n \), the case \( n = 1 \) following straight from the assumption that \( R \) is residually finite-dimensional. Now assume \( n > 1 \). Via induction, there is a finite-codimension ideal \( J \) of \( R \) such that \( a_2, \ldots, a_n \) are linearly independent over \( J \). Let \( I \) be an ideal of finite codimension in \( R \).

Assuming the lemma fails, \( I \cap J \) contains a linear combination \( \sum_{i=1}^{n} \lambda_i a_i \) with \( \lambda_1 \neq 0 \). This implies that every finite-codimension ideal of \( R \) contains some linear combination \( a_1 + \sum_{i=2}^{n} \mu_i a_i \). Since \( a_2, \ldots, a_n \) are linearly independent over \( J \), there are scalars \( \mu_2, \ldots, \mu_n \) such that \( a_1 + \sum_{i=2}^{n} \mu_i a_i \) belongs to every finite-codimension ideal of \( R \) contained in \( J \). But this means \( a_1 + \sum_{i=2}^{n} \mu_i a_i = 0 \), a contradiction.
**Lemma 2**

Let $R$ and $S$ be residually finite-dimensional $\mathbb{C}$-algebras. Let $0 \neq u \in R \otimes S$. There are finite-codimension ideals $I$ of $R$ and $S$ of $J$ such that $u \notin I \otimes J$. In particular, $R \otimes S$ is residually finite-dimensional.
Proof:

We may write $u$ as a sum $u = \sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n$ linearly independent elements of $R$ and $b_1, \ldots, b_n$ nonzero elements of $S$. Lemma 1 yields a finite-codimension ideal $I$ of $R$ with the property that $a_1, \ldots, a_n$ are linearly independent over $I$. Denote the natural homomorphism $R \to R/I$ by a bar. There are finite-codimension ideals $J_1, \ldots, J_n$ of $S$ satisfying $b_i \not\in J_i$ for $1 \leq i \leq n$. Letting $J = \bigcap_{i=1}^{n} J_i$ and $\bar{S} = S/J$, the elements $\bar{a}_1 \otimes \bar{b}_1, \ldots, \bar{a}_n \otimes \bar{b}_n$ are linearly independent in $\bar{R} \otimes \bar{S}$. 
The setup

**Proof of the theorem:**

Let \( R = \mathbb{C}G/V^\perp \) and \( S = \mathbb{C}H/W^\perp \). The group algebra \( \mathbb{C}[G \times H] \) is isomorphic to the tensor product \( \mathbb{C}G \otimes \mathbb{C}H \) acting on \( V \otimes W \) via \((x \otimes y)(v \otimes w) = xv \otimes yw\), where \( x \in \mathbb{C}G, y \in \mathbb{C}H, v \in V, w \in W \). If \( x \in V^\perp \) or \( y \in W^\perp \), then \( x \otimes y \) is the zero map, so that the above representation induces a \( R \otimes S \)-module structure on \( V \otimes W \). It follows that \( \mathbb{C}[G \times H]/(V \otimes W)^\perp \) is a homomorphic image of the ring \( R \otimes S \). If at least one of the modules \( V \) and \( W \) has finite dimension, then the mapping \( f \otimes g \) to the map \( x \otimes y \mapsto f(x)g(y) \) yields an isomorphism from \( V \otimes W \) into \( (R \otimes S)^* \).
**Proof of the theorem, continued:**

Let $0 \neq u \in R \otimes S$. As before, we write $u \sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n$ linearly independent elements of $R$ and $b_1, \ldots, b_n$ nonzero elements of $S$. Lemma 1 yields a finite-codimension ideal $I$ of $R$ with the property that $a_1, \ldots, a_n$ stay linearly independent over $I$. Let $R_1 = R/I$, $V_1 = R_1$. As remarked above, the dual space of $R_1 \otimes S$ is $V_1 \otimes W$, a $k = \dim R_1$-fold direct sum of isomorphic copies of $W$.

Let $J$ be a finite-codimension ideal of $S$ such that $b_i \notin J$ for all $i$. Then $u \notin I \otimes J$ and, with $I \otimes J$ is a finite-codimension ideal of $R \otimes S$ that does not contain $u$. 
Let $G$ be an abelian group and let $V$ be a variety on $G$. We say that *spectral synthesis holds on $V$* if every subvariety of $V$ is synthesizable. Let $R = \mathbb{C}G/V\perp$.

A commutative ring is called an *$N$-ring* if every ideal of the ring is contracted from a Noetherian extension ring.

**Theorem** (BW, 2019)

The variety $V$ has spectral synthesis if and only if, for every maximal ideal $M$ of $V$, $R/M \cong \mathbb{C}$ and the localisation of $R$ at $M$ is an $N$-ring.
It (eventually) follows from the just-mentioned theorem that, if spectral synthesis holds on $V$, there is, for every $n$ and every maximal ideal $M$ of $R$, a finite-dimensional subvariety $W$ of $V$ such that every finite dimensional indecomposable subvariety annihilated by $M^n$ is a homomorphic image of $W$. In particular, there is an absolute bound on the dimension of a finite-dimensional indecomposable subvariety in terms of its radical and radical length.
The question now is if the converse holds, i.e. if a variety on $G$ in which such a bound exists must have spectral synthesis.


