Tensor products of synthesizable modules and a question on varieties with spectral synthesis

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Let G be an abelian group and $\mathcal{C}G$ the space of functions $G \to \mathbb{C}$, endowed with the topology of pointwise convergence. By linear extension, $\mathcal{C}G$ emerges as isomorphic to the (algebraic) dual space of $\mathbb{C}G$ whereas, in turn, $\mathbb{C}G$ is isomorphic to the space of continuous linear functionals $\mathcal{C}G \to \mathbb{C}$ - the latter isomorphism effected by identifying an element u in $\mathbb{C}G$ with the function $f \mapsto f(u)$, where $f \in \mathbb{C}G^*$.
The group G acts on $\mathcal{C}G$ via $(xf)(g) = f(xg)$. The closed submodules of $\mathcal{C}G$ with respect to this action are called *varieties on* G. Thanks to the Hahn-Banach theorem, we have:

Theorem 1

If V is a variety on G, then V is isomorphic to the dual of $\mathbb{C}G/V^{\perp}$, where $V^{\perp}$ is the ideal $\{u \in \mathbb{C}G \,|\, h(u) = 0 \text{ for } h \in V\}$.

Let G and H be abelian groups. The group algebra $\mathcal{C}[G \times H]$ is isomorphic to the tensor product of $\mathbb{C}G$ with $\mathbb{C}H$. Let V and W be varieties on G and on H, respectively. Mapping a tensor $f \otimes g$ to the map $xy \mapsto f(x)g(y)$ ($x \in G$, $y \in H$) gives rise to a submodule of $\mathcal{C}[G \times H]$ that is isomorphic to $V \otimes W$, $G \times H$ acting on $V \otimes W$ via $xy(v \otimes w) = xu \otimes yv$. But is that submodule a variety?

Let $V = \mathcal{C}G$ and $W = \mathcal{C}H$. Let $0 \neq u \in \mathbb{C}G \otimes \mathbb{C}H$. The element u may be written as a sum $\sum_{i=1}^{n} a_i \otimes b_i$ with *linearly independent* elements $a_1, \ldots, a_n$ of V and nonzero elements $b_1, \ldots, b_n$ of W. Let $j \in \{1, \ldots, n\}$. There is $f \in \mathcal{C}G$ satisfying $f(a_j) = 1$ and $f(a_i) = 0$ if $i \neq j$. For $g \in \mathcal{C}H$, $\sum_{i=1}^{n} f(a_i)g(b_i) = f(a_j)g(b_j)$. So if $u \in (V \otimes W)^{\perp}$, then $g(b_j) = 0$ for all $g \in W$, which means that $b_j = 0$, a contradiction. It follows that the annihilator ideal $V \otimes W$ is the zero ideal. However, if both G and H are infinite, then the dual of $\mathbb{C}G \otimes \mathbb{C}H$ contains elements that are not linear combinations of tensors $f \otimes h$ with $f \in V$ and $h \in W$.

The closure $V \otimes W$ is a variety for the direct product $G \times H$. We outline a proof of the following

Theorem 2

If V and W are synthesizable, then so is the closure of $V \otimes W$.

A $\mathbb{C}$-algebra A is residually finite-dimensional if every nonzero element of A is nonzero in a finite quotient of A. A direct corollary of theorem 1 is that

A variety V on G is synthesizable (i.e. the closure of its finite-dimensional subvarieties) if and only if the ring $\mathbb{C}G/V^{\perp}$ is residually finite dimensional.

Lemma 1

Let R be a $\mathbb{C}$-algebra. If R is residually finite-dimensional, then finitely many linearly independent elements of R are linearly independent over some finite-codimension ideal of R.

Proof:

Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ linearly independent elements of R. We proceed via induction on n, the case $n = 1$ following straight from the assumption that R is residually finite-dimensional. Now assume $n > 1$. Via induction, there is a finite-codimension ideal J of R such that $a_2, \ldots, a_n$ are linearly independent over J. Let I be an ideal of finite codimension in R.

Assuming the lemma fails, $I \cap J$ contains a linear combination $\sum_{i=1}^{n} \lambda_i a_i$ with $\lambda_1 \neq 0$. This implies that every finite-codimension ideal of R contains some linear combination $a_1 + \sum_{i=2}^{n} \mu_i a_i$. Since $a_2, \ldots, a_n$ are linearly independent over J, there are scalars $\mu_2, \ldots, \mu_n$ such that $a_1 + \sum_{i=2}^{n} \mu_i a_i$ belongs to every finite-codimension ideal of R contained in J. But this means $a_1 + \sum_{i=2}^{n} \mu_i a_i = 0$, a contradiction.

Lemma 2

Let R and S be residually finite-dimensional $\mathbb{C}$-algebras. Let $0 \neq u \in R \otimes S$. There are finite-codimension ideals I of R and S of J such that $u \notin I \otimes J$. In particular, $R \otimes S$ is residually finite-dimensional.

Proof:

We may write u as a sum $u = \sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n$ linearly independent elements of R and $b_1, \ldots, b_n$ nonzero elements of S. Lemma 1 yields a finite-codimension ideal I of R with the property that $a_1, \ldots, a_n$ are linearly independent over I. Denote the natural homomorphism $R \to R/I$ by a bar. There are finite-codimension ideals $J_1, \ldots, J_n$ of S satisfying $b_i \notin J_i$ for $1 \leq i \leq n$. Letting $J = \bigcap_{i=1}^{n} J_i$ and $\bar{S} = S/J$, the elements $\bar{a}_1 \otimes \bar{b}_1, \ldots, \bar{a}_n \otimes \bar{b}_n$ are linearly independent in $\bar{R} \otimes \bar{S}$.

Proof of the theorem:
Let $R = \mathbb{C}G/V^{\perp}$ and $S = \mathbb{C}H/W^{\perp}$. The group algebra $\mathbb{C}[G \times H]$ is isomorphic to the tensor product $\mathbb{C}G \otimes \mathbb{C}H$ acting on $V \otimes W$ via $(x \otimes y)(v \otimes w) = xv \otimes yw$, where $x \in \mathbb{C}G$, $y \in \mathbb{C}H$, $v \in V$, $w \in W$. If $x \in V^{\perp}$ or $y \in W^{\perp}$, then $x \otimes y$ is the zero map, so that the above representation induces a $R \otimes S$-module structure on $V \otimes W$. It follows that $\mathbb{C}[G \times H]/(V \otimes W)^{\perp}$ is a homomorphic image of the ring $R \otimes S$. If at least one of the modules V and W has finite dimension, then the mapping $f \otimes g$ to the map $x \otimes y \mapsto f(x)g(y)$ yields an isomorphism from $V \otimes W$ into $(R \otimes S)^*$.

Proof of the theorem, continued:

Let $0 \neq u \in R \otimes S$. As before, we write $u \sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n$ linearly independent elements of R and $b_1, \ldots, b_n$ nonzero elements of S. Lemma 1 yields a finite-codimension ideal I of R with the property that $a_1, \ldots, a_n$ stay linearly independent over I. Let $R_1 = R/I$, $V_1 = R_1$. As remarked above, the dual space of $R_1 \otimes S$ is $V_1 \otimes W$, a $k = \dim R_1$-fold direct sum of isomorphic copies of W.

Let J be a finite-codimension ideal of S such that $b_i \notin J$ for all i. Then $u \notin I \otimes J$ and, with $I \otimes J$ is a finite-codimension ideal of $R \otimes S$ that does not contain u.

Let G be an abelian group and let V be a variety on G. We say that *spectral synthesis holds on* V if every subvariety of V is synthesizable. Let $R = \mathbb{C}G/V^{\perp}$.
A commutative ring is called an *N- ring* if every ideal of the ring is contracted from a Noetherian extension ring.

Theorem (BW, 2019)

The variety V has spectral synthesis if and only if, for every maximal ideal M of V, $R/M \cong \mathbb{C}$ and the localisation of R at M is an N-ring.

It (eventually) follows from the just-mentioned theorem that, if spectral synthesis holds on V, there is, for every n and every maximal ideal M of R, a finite-dimensional subvariety W of V such that every finite dimensional indecomposable subvariety annihilated by M^n is a homomorphic image of W.
In particular, there is an absolute bound on the dimension of a finite-dimensional indecomposable subvariety in terms of its radical and radical length.

The question now is if the converse holds, i.e. if a variety on G in which such a bound exists must have spectral synthesis.

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