Continuous Association Schemes and Hypergroups

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Debrecen, June, 2020

Motivation: Product formulas for spherical functions

Let (G, H) be a Gelfand pair.
Then the double coset space $X := G//H$ is a commutative hypergroup (Dunkl, Jewett, Spector).
Let $\varphi \in \mathcal{C}_b(X)$ be a hypergroup character, i.e.,continuous, bounded, multiplicative, and compatible with the involution.
Then φ is positive definite on X,
but not necessarily positive definite as a biinvariant function on G.

There are (G, H) where pointwise products of hypergroup characters are not longer positive definite, i.e., where no complete positive dual convolution exists: hyperbolic spaces (Flensted, Jensen, 1979), infinite distance transitive graphs (Voit, 2003).

On the other hand: If $\varphi \in supp\, \pi$, then φ is the a locally uniform limit of functions of the form $f * f^*$ for $f \in \mathcal{C}_c(X)$. Hence φ "is" a positive definite biinvariant function on G or a positive definite invariant kernel on G/H.

Thus: There is a positive dual convolution on $supp\, \pi$.

In many cases:

Spherical functions are special functions depending on continuous parameters.
For a discrete family of parameters, these functions are spherical functions.

The double coset convolutions can be computed explicitly.
They can be extended in an analytic way to hypergroup convolutions for a continuous range of parameters via a theorem of Carlson.

PROBLEM (in particular for symmetric spaces of higher rank):
The dual convolutions are much more complicated and often unknown.
Thus, analytic continuation does not work.
Therefore the existence of positive dual convolutions remains unclear beyond the group cases.

Are there further algebraic structures which lead to commutative hypergroup structures with dual positive convolutions on $supp\,\pi$?

Candidate from combinatorics: Finite commutative association schemes!

These objects generalize homogeneous spaces G/H associated with finite Gelfand pairs (G, H).

They always lead to finite commutative hypergroups
(which correspond to $G//H$ in the group setting),

and these finite commutative hypergroups always admit dual hypergroups.

Moreover, there are examples beyond groups.

Reference: Books of Bannai and Ito, Bailey, Zieschang; Online article: van Dam, Koolen, Tanaka: Distance regular graphs (2016)).

In this talk: Proposal of a generalization.

Classical finite commutative association schemes:

Definition

Let X, D be finite nonempty sets.
Let $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ $(i \in D)$ and with:

(1) There exists $e \in D$ with $R_e = \{(x,x) : x \in X\}$.

(2) There exists an involution $i \mapsto \bar{i}$ on D such that for $i \in D$, $R_{\bar{i}} = \{(y,x) : (x,y) \in R_i\}$.

(3) For all $i, j, k \in D$ and $(x,y) \in R_k$, the number

$$p_{i,j}^k := |\{z \in X : (x,z) \in R_i \text{ and } (z,y) \in R_j\}|$$

is independent of $(x,y) \in R_k$.

Then $\Lambda := (X, D, (R_i)_{i\in D})$ is called a finite association scheme with intersection numbers $(p_{i,j}^k)_{i,j,k\in D}$ and identity e.

For $i \in D$ form the adjacency matrices

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \in D,\ x,y \in X).$$

A_e is the identity, and $A_i^T = A_{\bar{i}}$ for $i \in D$.
Moreover, for $i,j \in D$, $A_i A_j = \sum_{k \in D} p_{i,j}^k A_k$.

Define the valencies $\omega_i := p_{i,\bar{i}}^e$ of R_i or $i \in D$.

Then the renormalized matrices $S_i := \frac{1}{\omega_i} A_i \in \mathbb{R}^{X \times X}$ are stochastic, i.e., all row sums are equal to 1. Moreover,

$$S_i S_j = \sum_{k \in D} \frac{\omega_k}{\omega_i \omega_j} p_{i,j}^k S_k \quad (i,j \in D),$$

and the linear span of the S_i is a finite dimensional algebra.
This algebra is isomorphic with the algebra of measures of some finite hypergroup structure on D.
An association scheme is called commutative if so is this hypergroup.

Generalization:

The product formulas for the stochastic matrices of finite association schemes can be extended to a topological setting with X, D as locally compact spaces,
where the stochastic matrices are replaced by Markov kernels,
and the finite summations by integrals over compacta.

For details see
M. Voit, Continuous Association Schemes and Hypergroups,
J. Austral. Math. Soc. 106 (2018), 361-426.

Axioms too complicated for this talk.

Thus restriction to the discrete case in this talk:

Discrete commutative association schemes:

Definition

Let X, D be at most countable sets.
Let $(R_i)_{i\in D}$ be a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$ with:

(1) There exists $e \in D$ with $R_e = \{(x,x) : x \in X\}$.

(2) There exists an involution $i \mapsto \bar{i}$ on D as before.

(3) For all $i, j, k \in D$ and $(x,y) \in R_k$, the number

$$p_{i,j}^k := |\{z \in X : (x,z) \in R_i \text{ and } (z,y) \in R_j\}|$$

is **finite** and independent of $(x,y) \in R_k$.

Then $\Lambda := (X, D, (R_i)_{i\in D})$ is called a discrete association scheme.

As before: There is an associated algebra generated by suitable stochastic matrices. This leads to a discrete hypergroup $(D, *)$.

Again, Λ is called commutative if so is $(D, *)$.

Examples:

Let (G, H) be a Gelfand pair where H is open in G.
Put $X := G/H := \{gH : g \in G\}$ and $D := G//H := \{HgH : g \in G\}$.
Then the partition

$$R_{HgH} := \{(xH, yH) \in X \times X : Hx^{-1}yH = HgH\} \quad (HgH \in D)$$

of $X \times X$ leads to a discrete commutative association scheme with identity HeH and involution $HgH \mapsto Hg^{-1}H$.
The associated hypergroup is the double coset hypergroup $G//H$.

Notice: There exist commutative examples beyond these group examples (distance regular graphs, buildings).

Generalized discrete commutative association schemes:

Definition

Let X, D be at most countable. Let $(R_i)_{i \in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$. Let $\tilde{S}_i \in \mathbb{R}^{X \times X}$ for $i \in D$ be stochastic matrices. Assume that:

(1) $(X, D, (R_i)_{i \in D})$ is a discrete association scheme.

(2) For all $i \in D$ and $x, y \in X$, $\tilde{S}_i(x, y) > 0$ if and only if $(x, y) \in R_i$.

(3) For all $i, j, k \in D$ there exist $\tilde{p}_{i,j}^k \geq 0$ with $\tilde{S}_i \tilde{S}_j = \sum_{k \in D} \tilde{p}_{i,j}^k \tilde{S}_k$.

(4) There exists a positive measure $\omega_X \in M^+(X)$ with $supp\, \omega_X = X$ and

$$\omega_X(\{y\}) \tilde{S}_{\bar{i}}(y, x) = \omega_X(\{x\}) \tilde{S}_i(x, y) \quad \text{for} \quad i \in D, \; x, y \in X.$$

Then $\Lambda := (X, D, (R_i)_{i \in D}, (\tilde{S}_i)_{i \in D})$ is called a generalized association scheme.

Λ is called commutative if $\tilde{S}_i \tilde{S}_j = \tilde{S}_j \tilde{S}_i$ for all $i, j \in D$.

Remarks

(1) Generalized discrete association scheme always are related to some discrete association scheme with the same spaces X, D!

(2) Each commutative discrete association scheme is also a generalized one in a canonical way.

(3) There exist examples of **infinite** generalized commutative discrete association schemes, which are not discrete association schemes (homogeneous trees where additional drifts are invented).

Proposition

Let Λ be a generalized commutative association scheme as above with deformed intersection numbers $\tilde{p}_{i,j}^k$. Then $\delta_i \tilde{*} \delta_j := \sum_{k \in D} \tilde{p}_{i,j}^k \delta_k$ leads to a discrete commutative hypergroup $(D, \tilde{*})$ with Haar measure

$$\Omega := \sum_{i \in D} \omega_i \delta_i \quad \textit{with} \quad \omega_i := \frac{1}{\tilde{p}_{i,\bar{i}}^e} > 0 \quad (i \in D).$$

AIM: Let $(D, \tilde{*})$ be a discrete commutative hypergroup which is associated with some generalized commutative association scheme. Then there exists a positive dual convolution on the support of the Plancherel measure.

In the proof an additional condition (T) appears which relates the usual convolutions on $(D, \tilde{*})$ with some canonical convolution operators on X.

Theorem

If Λ is a generalized discrete commutative association scheme with this translation property (T).
Consider the associated discrete commutative hypergroup $(D, \tilde{*})$.
Then, there exists a positive product formula for all characters of $(D, \tilde{*})$ which are contained in $\operatorname{supp} \pi$.

Here are some essential results on this translation property (T):

Theorem

If $\Lambda := (X, D, (R_i)_{i \in D})$ is a discrete commutative association scheme, then (T) holds. In particular, In the setting of Gelfand pairs, (T) holds.

Theorem

If Λ is a finite generalized commutative association scheme, then (T) holds.

Moreover:

Theorem

If Λ is a finite generalized commutative association scheme with (T), then Λ is a classical finite commutative association scheme.

Corollary

Each finite generalized commutative association scheme is a classical one.

Negative message: In the finite case, the generalized theory does not lead to any new examples.

Positive message: In the finite case, the existence of an algebra of stochastic matrices in the generalized theory already ensures, that these matrices admit some integrality conditions which are inherent from the partitions of the finite commutative association scheme.

This rigidity was completely unexpected for me when starting the subject.

On the other hand: in the infinite case, there exist examples beyond classical discrete commutative association schemes.
For instance, constructions like hypergroup joins can be transfered to (generalized) association schemes.

Many of the preceding results can be extended from the discrete case to the general, continuous case.

Thank you for your attention!