Symmetry problems in harmonic analysis

Alexander G. Ramm

Mathematics Department,
Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@ksu.edu
http://www.math.ksu.edu/ ∼ ramm

Abstract

Symmetry problems in harmonic analysis are formulated and solved. One of these problems, Theorem A, is equivalent to the refined Schiffer's conjecture, the other, Theorem B, is equivalent to the refined Pompeiu problem. Both were solved by the author.
Let $k = const > 0$ be fixed, S^2 be the unit sphere in $\mathbb{R}^3$, D be a connected bounded domain with C^2-smooth boundary S, $j_0(r)$ be the spherical Bessel function.
The harmonic analysis symmetry problems are stated in the following theorems.
Theorem A. *Assume that* $\int_S e^{ik\beta\cdot s}ds = 0$ *for all* $\beta \in S^2$. *Then* S *is a sphere of radius* a, *where* $j_0(ka) = 0$.
Theorem B. *Assume that* $\int_D e^{ik\beta\cdot x}dx = 0$ *for all* $\beta \in S^2$. *Then* D *is a ball of radius* a, *where* $j_0'(ka) = 0$.

Introduction

Symmetry problems for PDE were studied in many publications by many authors, see, [1], [2] and references therein.
Throughout we assume that D is a bounded connected C^2-smooth domain in $\mathbb{R}^3$, S is the boundary of D, N is the unit normal to S, pointing out of D, u_N is the normal derivative of u on S, $D' = \mathbb{R}^3 \setminus D$, S^2 is the unit sphere in $\mathbb{R}^3$, $k > 0$ and c are fixed constants, $j_0(r)$ is the spherical Bessel function and $g(x,y,k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$.
By $(u_N)_-$ we denote the limiting value on S of u_N from D' and by $(u_N)_+$ we denote the limiting value on S of u_N from D. We use the following formula:

$$g(x,y,k) = \frac{e^{ik|x|}}{4\pi|x|} e^{ik\beta\cdot y} + O(\frac{1}{|x|^2}), \quad |x| \to \infty, \quad \beta = -x/|x|, \qquad (2.1)$$

where $|y| \le R$, $R > 0$ is an arbitrary large fixed number.

Problem formulation 1

We reduce the symmetry problem in harmonic analysis to a symmetry problem for PDE, [2], [3]. Theorem 3.1, that we use, was proved in [2] and in [3]. Let us formulate it. The assumptions about D are the same as in this paper. Below c_j, $j = 0, 1, 2$, are some constants.

Theorem 3.1. *Assume that the problem*

$$(\nabla^2 + k^2)w = c_0 \quad in \quad D, \quad u|_S = c_1, \quad u_N|_S = c_2, \tag{2.2}$$

is solvable. If

$$|c_1 - c_0 k^{-2}| + |c_2| > 0, \tag{2.3}$$

then D is a ball.

In [2], [3] the refined Schiffer's conjecture (SC) is proved. Let us formulate this result.

Theorem 1. *Assume that*

$$\Delta u + k^2 u = 0 \quad in \quad D, \quad u|_S = 0, \quad u_N = c. \tag{2.4}$$

Then S is a sphere of radius a such that $j_0(ka) = 0$.

Problem formulation 2

Let us formulate our new result: formulation and solution of a symmetry problem in harmonic analysis (Problem HA):

Theorem A. *Assume that*

$$\int_S e^{ik\beta\cdot y}dy = 0 \quad \forall\beta \in S^2. \tag{2.5}$$

Then S is a sphere of radius a, where $j_0(ka) = 0$.

Theorem B. *Assume that*

$$\int_D e^{ik\beta\cdot x}dx = 0 \quad \forall\beta \in S^2, \tag{2.6}$$

where D is a bounded connected domain in $\mathbb{R}^3$ and S^2 is the unit sphere in $\mathbb{R}^3$. Then D is a ball of radius a, where $j_0'(ka) = 0$.

We prove that the harmonic analysis symmetry problem (HA), Theorem A, is equivalent to the refined Schiffer's conjecture (SC), Theorem 1: if Theorem A holds then Theorem 1 holds and vice versa. Theorem B is equivalent to Pompeiu problem. The author does not know any symmetry results in harmonic analysis of the type presented in Theorems A and B.

Problem formulation 3

If problem (2.4) has a solution then this solution is unique by the uniqueness of the solution to the Cauchy problem for the Helmholtz elliptic equation (2.4).
The solution to equation (2.4) by the Green's formula is:

$$u(x) = c\int_S g(x,t)dt, \quad x \in D; \quad u(x) := c\int_S g(x,t)dt = 0, \quad x \in D'. \qquad (2.7)$$

These formulas are obtained by the standard application of the Green's formula. Namely, one starts with the equations

$$(\nabla_y^2 + k^2)u = 0, \qquad (2.8)$$

$$(\nabla_y^2 + k^2)g(x,y) = -\delta(x-y), \quad y \in D. \qquad (2.9)$$

Multiply (2.8) by $g = g(x,y)$, equation (2.9) by $u(y)$, subtract the second equation from the first, integrate over D and use the definition of the delta-function $\delta(x-y)$ and the boundary conditions in (2.4) to get equation (2.7).

Proofs 1

The function u, defined by the first formula (2.7) in $\mathbb{R}^3$ satisfies the radiation condition

$$u_r - iku = O(|x|^{-2}), \quad r := |x| \to \infty \tag{2.10}$$

uniformly with respect to directions of x.
Let $B_R = \{x : |x| \leq R\}$, $D \subset B_R$. If D is a ball B_a of radius a, and u solves (2.4) then a solves the equation $j_0(ka) = 0$, and the solution u has the form:

$$u = c\frac{j_0(kr)}{kj_0'(ka)}, \quad r = |x|, \tag{2.11}$$

where $j_0'(r) := \frac{dj_0(r)}{dr}$.

Proofs 2

Proof of Theorem A.
Assume that (2.5) holds. Let u be defined by the first formula (2.7) in $\mathbb{R}^3$. Then, due to (2.5), one has:

$$u = O(|x|^{-2}), \quad |x| \to \infty. \tag{2.12}$$

Moreover, u, defined in (2.7), solves the equation

$$(\nabla^2 + k^2)u = 0 \quad in \quad D'. \tag{2.13}$$

By the known lemma, see, for example, [1], p.30, Lemma 1.2.1, it follows from (2.12) and (2.13) that $u = 0$ in D'. Let us formulate the lemma we have used.
Lemma 1. *If* (2.12) *and* (2.13) *hold, then* $u = 0$ *in* D'.
Since $u = 0$ in D', u is a single layer potential and S is C^2-smooth, one concludes that u is continuous up to S together with its first derivatives, so

$$u = 0, \quad (u_N)_- = 0 \quad on \quad S. \tag{2.14}$$

Proofs 2

Continuation of Proof of Theorem A.
By the jump formula for the normal derivative of u (see, for example, [1], p. 18), one gets:

$$(u_N)_+ - (u_N)_- = (u_N)_+ = 1, \tag{2.15}$$

since the density of the single layer potential u is equal to 1 and $(u_N)_- = 0$. Therefore, if (2.5) holds, then u solves problem (2.4) with $c = 1$. Consequently, by Theorem 1, S is a sphere of radius a, where $j_0(ka) = 0$.
Theorem A is proved. □

Proofs 4

Lemma 2. *Theorem 1 and Theorem A are equivalent.*

Proof of Lemma 2. Assume that Theorem 1 holds. Define u by formula (2.7). As $|x| \to \infty$, $x/|x| = -\beta$, this yields (2.5). So, if Theorem 1 holds, then Theorem A holds.

Conversely, Suppose that Theorem A holds. From (2.5) one derives the relation:

$$u(x) := \int_S g(x,t)dt = 0 \quad in \quad D'. \tag{2.16}$$

Indeed, the integral $u(x)$ in (2.16) satisfies differential equation (2.13) in D' and $u = O(|x|^{-2})$ as $|x| \to \infty$. So $u = 0$ in D' by Lemma 1. Equation (2.13) for u holds in D, $u = 0$ on S by continuity, and $(u_N)_+ = 1$ on S by the jump formula for the normal derivatives of the single layer potential u. Thus, u solves problem (2.4). So, Theorem A yields the conclusion of Theorem 1.

Lemma 2 is proved. □

Proofs 5

Proof of Theorem B. Assume that (2.6) holds. Define

$$w(x) := \int_D g(x,t)dt, \quad x \in \mathbb{R}^3. \tag{2.17}$$

Then

$$(\nabla^2 + k^2)w = 0 \quad in \quad D', \tag{2.18}$$

and

$$w = O(|x|^{-2}) \quad as \quad |x| \to \infty. \tag{2.19}$$

Therefore, by Lemma 1, one concludes that

$$w = 0 \quad in \quad D'. \tag{2.20}$$

Since w is a volume potential which is continuous together with its first derivatives in $\mathbb{R}^3$, one gets from (2.20) and (2.17) that

$$w = 0, \quad w_N = 0 \quad on \quad S, \tag{2.21}$$

and

$$(\nabla^2 + k^2)w = -1 \quad in \quad D. \tag{2.22}$$

We now use Theorem 3.1 and conclude that D is a ball. Its radius a solves the equation $j_0'(ka) = 0$.
Theorem B is proved.

Proofs 5

The Pompeiu problem: assume that $f \in L^1_{loc}(\mathbb{R}^3)$ and

$$\int_D f(y+Gx)dx = 0 \quad \forall y \in \mathbb{R}^3, \forall G \in SO(2). \tag{2.23}$$

Does this imply $f = 0$? In 1929 Pompeiu published a paper in which he claimed that (2.23) implies $f = 0$. In 1944 a counterexample was published by L.Chakalov. In [2] the author proved that (2.23) implies that f is not zero if and only if D is a ball.

References

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