Sets of p-restriction and p-spectral synthesis

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Let $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ (Fourier transform) is a continuous function on $\mathbb{R}^n$, so if E is a subset of $\mathbb{R}^n$,

$$\hat{f}|_E \text{ is well-defined.}$$

If $1 < p < 2$ and $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^{p'}(\mathbb{R}^n), (\frac{1}{p} + \frac{1}{p'} = 1)$.

Question: Can we restrict $\hat{f}$ to $E \subseteq \mathbb{R}^n$ if E has Lebesgue measure zero? Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz functions on $\mathbb{R}^n$ and define $\mathcal{R}_E \colon \mathcal{S}(\mathbb{R}^n) \to C(E)$ by

$$\mathcal{R}_E(f) = \hat{f}|_E.$$

The Restriction Problem

Question: For what values of p and q does there exists a constant $C(n,p)$ for which

$$\|\hat{f}\|_{L^q(E)} \leq C(n,p)\|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$? In other words can we extend $\mathcal{R}_E$ to a continuous operator from $L^p(\mathbb{R}^n) \to L^q(E)$? (Hard Problem!)

Definition: We call a closed set E a set of *p-restriction* if there exists $C(n,p)$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|\hat{f}\|_{L^1(E)} \leq C(n,p)\|f\|_p \, (q = 1)$$

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Spectral Synthesis (Classical Case)

Let

$$I(E) = \{f \mid f \in L^1(\mathbb{R}^n), \hat{f} = 0 \text{ on } E\}$$

$$j(E) = \{f \mid f \in L^1(\mathbb{R}^n), \hat{f} = 0 \text{ on a nhd of } E\}.$$

Set $J(E) = \overline{j(E)}$. Both $I(E)$ and $J(E)$ are closed ideals in $L^1(\mathbb{R}^n)$.
If $I(E) = J(E)$, then we say E satisfies spectral synthesis.
Let $p \in (1, 2)$ and suppose E is a set of p-restriction. Define

$$I^p(E) = \{f \in L^p(E) \mid \hat{f}|_E = 0\}.$$

Remark: $I^p(E)$ is a closed $L^1(\mathbb{R}^n)$-submodule of $L^p(\mathbb{R}^n)$.

Set

$$k^p(E) = \{f \in \mathcal{S}(\mathbb{R}^n) \mid \hat{f} \text{ vanishes on a nhd of } E\}.$$

If $\overline{k^p(E)} = I^p(E)$, then we say that E is a set of *p-spectral synthesis*.

Theorem: Let $2 \leq n \in \mathbb{Z}$ and let E be a smooth compact hypersurface in $\mathbb{R}^n$ with constant relative nullity $\nu, 0 \leq \nu \leq n-2$. If E is a set of p-restriction for some p that satisfies one of the following:

1. $\frac{2(n-\nu)}{n+3-\nu} \leq p < 2$ and $0 \leq \nu < n-3$
2. $1 < p < 2$ for $n-3 \leq \nu \leq n-1$,

then E is a set of p-spectral synthesis.

The proof relies on the fact if E satisfies the hypothesis of the theorem then it has the p'-approximate property (Kanghui Guo 1989, 1995).

If $E = S^{n-1}$, the unit sphere in $\mathbb{R}^n$, then $\nu = 0$. Schwartz proved in the 1950's that S^{n-1} is **not** a set of spectral synthesis if $n \geq 3$.

Corollary: S^2 is a set of p-spectral synthesis for $1 < p \leq \frac{4}{3}$, and for $n \geq 4, S^{n-1}$ is a set of p-spectral synthesis for $\frac{2n}{n+3} \leq p \leq \frac{2n+2}{n+3}$.

Remark The lower bound is sharp and we suspect that the upper bound is $\frac{2n}{n+1}$.

Span of translates

Theorem: Let $1 < p < 2$ and let E be a compact subset of $\mathbb{R}^n$ with induced measure $d\sigma$. If E is a set of p-restriction, then $I^p(E) \neq L^p(\mathbb{R}^n)$.

Recall that if $f \in L^p(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, the translate of f by y is the function $f_y(x) = f(x - y)$.

Denote by $T^p[f]$ the closed subspace of $L^p(\mathbb{R}^n)$ spanned by f and its translates.

For $f \in L^1(\mathbb{R}^n)$ the zero set of f is $Z(f) = \{\eta \in \mathbb{R}^n \mid \hat{f}(\eta) = 0\}$.

We now obtain

Corollary: Let $1 < p < 2$ and let E be a compact subset of $\mathbb{R}^n$. If there exists an $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for which $E \subseteq Z(f)$ and $T^p[f] = L^p(\mathbb{R}^n)$, then E is not a set of p-restriction.

Theorem: Let E be a smooth compact submanifold of codimension k in $\mathbb{R}^n$. If there exists $f \in C_c(\mathbb{R}^n)$ such that $\hat{f}$ vanishes on E, then E is not a set of p-restriction for $\frac{2n}{n+k} \leq p \in \mathbb{R}$.

Theorem: Let $2 \leq n \in \mathbb{Z}$ and let E be a smooth compact hypersurface in $\mathbb{R}^n$ with constant relative nullity ν, for $0 \leq \nu \leq n-2$. If $\frac{2(n-\nu)}{n-\nu+1} \leq p \in \mathbb{R}$, then E is not a set of p-restriction

Thank you

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