An interplay between Gabor bases and Fuglede Conjecture

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Background

Let $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$. Let $\Lambda \subset \mathbb{R}^{2d}$ be a discrete and countable set.

The Gabor system (Gabor '46)

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i \langle \lambda, x \rangle} g(x - t) : (t, \lambda) \in \Lambda\}.$$

Gabor orthonormal bases

1. mutually orthogonal

$$\int e^{2\pi i \langle \lambda_1 - \lambda_2, x \rangle} g(x - t_1) \overline{g(x - t_2)} dx = 0.$$

   For $(t_1, \lambda_1) \neq (t_2, \lambda_2) \in \Lambda$.

2. Completeness

$$\|f\|^2 = \sum_{(t,\lambda) \in \Lambda} |\langle f, e^{2\pi i \langle \lambda, \cdot \rangle} g(\cdot - t) \rangle|^2, \quad f \in L^2(\mathbb{R}^d).$$

Definition: We call g a window function, and Λ a time-frequency shift set.
Applications: Gabor filters in image processing (e.g. fingerprint recognition), in communication and pattern analysis

Gabor orthonormal bases (GONB)

One of the fundamental problem in Gabor analysis:

Given $g \in L^2(\mathbb{R}^d)$, determine set

$$\{\Lambda : \mathcal{G}(g, \Lambda) \text{ is an GONB}\}.$$

Most trivial example: $g = \chi_{[0,1]^d}$, $\Lambda = \mathbb{Z}^d \times \mathbb{Z}^d$

1. Full characterization for $g = \chi_{[0,1]}$, $d = 1$ (Gabardo, Lai, Wang 2015)

$$\left\{ \Lambda = \bigcup_{j \in \mathbb{Z}} (\{j\} \times (\mathbb{Z} + a_j)) : a_j \in [0,1), \mathcal{G}(\chi_{[0,1]}, \Lambda) \text{ is an ONB} \right\}.$$

2. If g belongs to the modulation space $M^1(\mathbb{R}^d)$

$$\{\Lambda : \mathcal{G}(g, \Lambda) \text{ is an ONB}\} = \emptyset.$$

(Ascensi, Feichtinger and Kaiblinger, 2014. Partial result in the past used Balian-Low theorem for lattice)

Gabor orthonormal bases

What if $|g| = |K|^{-1/2}\chi_K$?

Proposition (Liu, Wang, 2003)

Let $|g| = |K|^{-1/2}\chi_K$*, and* $K \subset \mathbb{R}^d$ *is measurable with* $0 < |K| < \infty$*. Suppose that*

- K *tiles* $\mathbb{R}^d$ *by a discrete countable set* $\mathcal{J}$*. (K tiles by translations)*
- *For each* $t \in \mathcal{J}$*, the set of exponentials* $\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda_t\}$ *is an orthonormal basis for* $L^2(K)$*. (K is a spectral set)*

Take

$$\Lambda := \bigcup_{t\in\mathcal{J}} \{t\} \times \Lambda_t.$$

Then $\mathcal{G}(g,\Lambda)$ *is a Gabor orthonormal basis for* $L^2(\mathbb{R}^d)$*.*

Definition (tiling)

We say that $K \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ by a discrete countable set $\mathcal{J} \subset \mathbb{R}^d$, if

$$\sum_{t \in \mathcal{J}} \chi_K(x - t) = 1 \quad a.e.\ x \in \mathbb{R}^d$$

Definition (spectral)

We say that K is a spectral set if there is a countable set $\Lambda \subset \mathbb{R}^d$ such that the exponential functions

$$\mathcal{E}(\Lambda) = \{|K|^{-1/2} e^{2\pi i \langle x, \lambda \rangle} : \lambda \in \Lambda\}$$

is an orthonormal basis for $L^2(K)$.

Fuglede conjecture (1974): A set tiles by translations if and only if the set is spectral.

Fuglede Conjecture

Fuglede conjecture (1974): A set tiles by translations if and only if the set is spectral.

- Disproved later by Tao in 2003 in $\mathbb{R}^5$ and higher, and by other mathematicians (Kolountzakis, Matolcsi, etc.) in dimensions $d = 3, 4$.
- Iosevich and Pedersen; Lagarias, Reeds and Wang (2000) Tiling set and spectrum for $L^2([0,1]^d)$ is equivalent.
- Laba proves the conjecture for union of two interval (2001)
- For the union of three intervals (Bose, Kumar, Krishnan, Madan 2010)
- For convex domains in all dimensions (Lev, Matolcsi 2019)
- $\cdots$
- The conjecture holds in $\mathbb{Z}_{p^n}$ (Laba 2002), $\mathbb{Z}_p^2$ (Iosevich, M., Pakianathan 2016), $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ (Kiss, Somlai 2020), $\mathbb{Z}_{p^n q}$ (Malikiosis, Kolountzakis 2017), $\cdots$
- For group of p-adic numbers, $\mathcal{Q}_p$, p prime. (Fan at el. 2015)
- $\cdots$
- The conjecture is still open in dimensions $d = 1$ and 2.

Fuglede-Gabor problem

Problem (Fuglede-Gabor problem)

Suppose that $\mathcal{G}(|K|^{-1/2}\chi_K, \Lambda)$ is a GONB for some countable set $\Lambda \subset \mathbb{R}^{2d}$, is that true that K is a spectral set and tiles by translations?

- *Easy to show:* Take $g = \chi_K$ and $\Lambda = \mathcal{J} \times \Gamma$ (separable).

  $\mathcal{G}(g, \mathcal{J} \times \Gamma)$ is an ONB iff K tiles by $\mathcal{J}$ and is a spectral set by Γ.
- This rules out many examples of set K since Fuglede conjecture fails in general.
- On Gabor orthonormal bases over finite prime fields (A. Iosevich, M. Kolountzakis, Yu. Lyubarskii, A. Mayeli, J. Pakianathan, arXiv:1712.09120, to appear, Bulletin of the London Math. Society, 2020.)

Fuglede-Gabor problem

Conjecture (Liu-Wang, 2003)

Suppose that $\mathcal{G}(g, \mathcal{J} \times \Gamma)$ forms a GONB and supp$(g) = K$ is compact. Then

1. *K is a spectral set with spectrum Γ.*
2. *K is a translational tile with tiling set $\mathcal{J}$*
3. $|g| = |K|^{-1/2}\chi_K$.

1. (Liu-Wang, 2003) True if supp$(g) = [0, 1]$.
2. (Dutkay and Lai 2014) True if $g \geq 0$.
3. Conjecture is still open for general g with compact support.

GONB vs. Fuglede Conjecture

1. (Agora, Antezana, Kolountzakis, 2017) Fuglede-Gabor problem is true for union of two intervals.

   compare with: (Laba, 2000) Fuglede conjecture is true for union of two intervals.

2. (Iosevich and M., 2017) $K \subset \mathbb{R}^d$ is a convex set has a point of positive Gaussian curvature and $d \neq 1 \pmod 4$. Then K cannot be a GONB set.

   compare with: (Iosevich,Katz, Tao, 2001) $K \subset \mathbb{R}^d$ is a convex set has a point of positive Gaussian curvature, K is not a spectral set.

3. (Chung, Lai, 2017) K is a non-symmetric convex polytope. Then K cannot be a GONB set.

   compare with: (Kolountzakis, 2000) K is a non-symmetric convex body . Then K is not a spectral set.

Fuglede-Gabor problem

Theorem (Fuglede 1974)

K is a spectral set with spectrum a lattice if and only if K is a translational tile with the dual lattice.

Recall that $\Gamma = A(\mathbb{Z}^d)$ and then the dual lattice $\Gamma^\perp = A^{-T}(\mathbb{Z}^d)$.

Is F-G problem also solvable for lattices?

The following is a joint work with Chun-Kit Lai.

C.K. Lai, A. M., *Non-separable lattice, Gabor orthonormal bases and tiling*, Journal of Fourier Analysis and Applications volume 25, (2019)

- Most of our results are for lower-triangular lattices.
- A lattice $\Lambda \subset \mathbb{R}^{2d}$ is defined by an invertible block matrix:

$$\Lambda = M(\mathbb{Z}^{2d}) = \begin{pmatrix} A & D \\ C & B \end{pmatrix} (\mathbb{Z}^{2d}).$$

- For $\mathcal{G}(g, \Lambda)$ to be a GONB, the density of Λ must be equal to 1.
- We say the lattice Λ is lower-triangular if $D = 0$.
- An observation: Any rational lattice Λ can be expressed as a lower-triangular rational matrix:

$$\Lambda = \begin{pmatrix} A & O \\ C & B \end{pmatrix} (\mathbb{Z}^{2d}). \tag{1}$$

- Therefore, in our study, we first and mostly focus on lower-triangular lattices with density $dens(\Lambda) = 1$:

$$\Lambda = \bigcup_{m \in \mathbb{Z}^d} \{(Am, Cm + Bn) : n \in \mathbb{Z}^d\}, \text{and } |\det(AB)| = 1$$

Key lemma

Lemma (Lai and M.)

Let $\Lambda = M(\mathbb{Z}^{2d})$ *with* M *an* $2d \times 2d$ *invertible lower triangular block matrix of the form (1). Suppose that* $\mathcal{G}\left(|K|^{-1/2}\chi_K, \Lambda\right)$ *is a Gabor orthonormal basis. Then*

$$K = \bigcup_{j=1}^{N} D_j = \bigcup_{j=1}^{N} E_j \quad \text{for some } N \geq 1$$

where D_j*'s are almost disjoint fundamental domains of* $B^{-T}(\mathbb{Z}^d)$ *and* E_j*'s are almost disjoint fundamental domain of* $A(\mathbb{Z}^d)$*.*

Notice, as a result K multi-tiles $\mathbb{R}^d$ simultaneously by $A(\mathbb{Z}^d)$ and $B^{-T}(\mathbb{Z}^d)$.

Sketch of Proof for Key Lemma

- Recall that

$$\Lambda = \bigcup_{m \in \mathbb{Z}^d} \{(Am, Cm + Bn) : n \in \mathbb{Z}^d\}$$

- Take $m = 0$, then mutual orthogonality means that $B(\mathbb{Z}^d)$ is a mutual orthogonal set for $L^2(K)$. This implies K is a union of fundamental domain for lattice $B^{-T}(\mathbb{Z}^d)$ (by Jorgensen and Pedersen).
- (Lemma: Ron-Shen Duality) $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis if and only if $\mathcal{G}(g, \Lambda^\circ)$ is a Gabor orthonormal basis.

$$\Lambda^\circ = \begin{pmatrix} O & -I \\ I & O \end{pmatrix} (M^{-T})(\mathbb{Z}^{2d}) = \begin{pmatrix} B^{-T} & O \\ D & A^{-T} \end{pmatrix} (\mathbb{Z}^{2d}).$$

  Or

$$\Lambda^\circ = \bigcup_{m \in \mathbb{Z}^d} \{(B^{-T}m, Dm + A^{-T}n) : n \in \mathbb{Z}^d\}$$

- $\det(A) = \det(B^{-T})$, then the union of fundamental domains are same. That is

$$K = \bigcup_{j=1}^{N} D_j = \bigcup_{j=1}^{N} E_j.$$

Some remarks.

- Recall

$$K = \bigcup_{j=1}^{N} D_j = \bigcup_{j=1}^{N} E_j.$$

- If $N = 1$, then K is a common fundamental domain for both $A(\mathbb{Z}^d)$ and $B^{-T}(\mathbb{Z}^d)$. Hence, K is a translational tile and a spectral set.
- (Han and Wang, 2000) Given any matrix C and D such that

$$|\det(C)| = |\det(D)|,$$

there exists a common fundamental domain for both $C(\mathbb{Z}^d)$ and $D(\mathbb{Z}^d)$.

Lower triangular (LT) lattice

Theorem (F-G problem for LT lattices)

Let K be a subset of $\mathbb{R}^d$ with finite positive measure, and let $\Lambda \subset \mathbb{R}^{2d}$ be a lower triangular lattice.

$$\Lambda = \begin{pmatrix} A & O \\ C & B \end{pmatrix} (\mathbb{Z}^{2d}).$$

Suppose that $\mathcal{G}(|K|^{-1/2}\chi_K, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$ and $B = A^{-T}$. Then $N = 1$, hence K tiles and is spectral.

$$B = A^{-T} \Longrightarrow \Lambda = \begin{pmatrix} A & O \\ C & A^{-T} \end{pmatrix} (\mathbb{Z}^{2d}) \Longrightarrow \widetilde{\Lambda} = \begin{pmatrix} I & O \\ \widetilde{C} & I \end{pmatrix} (\mathbb{Z}^{2d})$$

Proposition

Suppose K is bounded and multi-tiles $\mathbb{R}^d$ with respect to $\mathbb{Z}^d$ at level N. That is

$$\sum_{n\in\mathbb{Z}^d} \chi_K(x-n) = N \quad a.e. x \in \mathbb{R}^d \; .$$

If $N > 1$, then there exists $\mathbf{m} \in \mathbb{Z}^d$ such that

1. $K \cap (K + \mathbf{m})$ *has positive Lebesgue measure.*
2. $K \cap (K + \mathbf{m})$ *consists of distinct representative (mod $\mathbb{Z}^d$).*
3. $K \cap (K + \mathbf{m})$ *is a packing by $\mathbb{Z}^d$, i.e.*

$$\sum_{n\in\mathbb{Z}^d} \chi_{K\cap(K+\mathbf{m})}(x+n) \leq 1. \; a.e. \; x \in \mathbb{R}^d$$

A complete solution to rational matrix on $\mathbb{R}^1$

Theorem (complete solution of F-G problem for rationals in D-1)

Suppose that Λ is a rational lattice in $\mathbb{R}^2$ with $dens(\Lambda) = 1$. That is

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\mathbb{Z}^2), \text{ with } ad - bc = 1$$

Suppose that $K \subset \mathbb{R}$ is bounded with positive measure.
If $\mathcal{G}(|K|^{-1/2}\chi_K, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R})$, then K tiles and is spectral.

Sketch of the proof:

$$\Lambda \Longrightarrow \Lambda = \begin{pmatrix} \alpha & 0 \\ \beta & 1/\alpha \end{pmatrix} (\mathbb{Z}^2) \Longrightarrow \text{follows by the theorem for LT lattices}$$

Lower triangular matrix

Does a multi-tiling set generates a Gabor ONB?

Example (K multi-tiles and generates GONB)

Let $K = [0,2] \times [0,1]$ and $\Lambda = \begin{pmatrix} I & O \\ C & B \end{pmatrix} (\mathbb{Z}^4)$ where $B = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. The system $\mathcal{G}(|K|^{-1/2}\chi_K, \Lambda)$ is a Gabor orthonormal basis if c_{21} is an odd number.

Lower triangular matrix

Example (K multi-tiles and DOES NOT generate GONB)

Let $K = [0,2]^2$ and let B be the matrix as in the previous Example. Then K is union of 4 fundamental domains of the lattice $\mathbb{Z}^2$ and union of 4 fundamental domains of $B^{-T}(\mathbb{Z}^2)$. However, there is no rational matrix C for which χ_K is a window function for the lower triangular lattice $\Lambda = \begin{pmatrix} I & O \\ C & B \end{pmatrix}(\mathbb{Z}^4)$.

Upper triangular (UT) lattices

Definition

We say a lattice $\Lambda = M(\mathbb{Z}^{2d})$ is an upper triangular lattice, if M is an upper triangular block matrix.

Theorem (F-G problem for UT lattices)

Suppose that $K \subset \mathbb{R}^d$ with positive finite measure. Suppose that $\Lambda \subset \mathbb{R}^{2d}$ is a lattice such that $\Lambda = \begin{pmatrix} A & D \\ O & A^{-t} \end{pmatrix} (\mathbb{Z}^{2d})$, $A^{-1}D$ symmetric rational matrix. If $\mathcal{G}(|K|^{-1/2}\chi_K, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$, then K tiles and is spectral.

Lemma (Auxilary lemma)

For $\Lambda = \begin{pmatrix} I & D \\ O & I \end{pmatrix} (\mathbb{Z}^{2d})$, with D rational and symmetric, there are integer matrices E and X such that $\Lambda = \begin{pmatrix} E^{-t} & 0 \\ X & E \end{pmatrix} (\mathbb{Z}^{2d})$.

Upper triangular matrix

Example

Let $\mathcal{O}_8$ be the octagon symmetrically centred at the origin with integer vertices. $\mathcal{O}_8$ multi-tiles $\mathbb{R}^2$ with $\mathbb{Z}^2$ and it is the union of $s = 14$ fundamental domains of $\mathbb{Z}^2$. Yet there doesn't exist any upper triangular lattice Λ for which $\mathcal{G}(\mathcal{O}_8, \Lambda)$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.

Open questions

General Lattices: $\Lambda = \begin{pmatrix} A & D \\ C & B \end{pmatrix} (\mathbb{Z}^{2d})$.

1. **Rational case:** $\Lambda = \begin{pmatrix} I & O \\ C & B \end{pmatrix} (\mathbb{Z}^{2d})$, where B and C are $d \times d$ rational matrices but B is not necessarily an integral matrix.
2. **Irrational case:** Λ is a lattice of the form of $\begin{pmatrix} I & D \\ C & B \end{pmatrix} (\mathbb{Z}^{2d})$ which contains irrational entries.

   Han and Wang's Conjecture: *Let* $\Lambda = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} (\mathbb{Z}^{2})$ *where* α *is irrational. Then there doesn't exist compactly supported window* g *such that* $\mathcal{G}(g, \Lambda)$ *forms a Gabor orthonormal basis for* $L^2(\mathbb{R})$.
3. **Exponentialy completeness:** Given K, determine set Γ such that $\{e^{2\pi i \langle x, \gamma \rangle} : \gamma\}$ is exponentially complete:

$$\nexists\ a \in \mathbb{R}^d \setminus \Gamma \quad \text{such that} \quad \hat{\chi}_K(a - \gamma) = 0 \quad \forall \gamma$$

(Item 3 is an ongoing project with Alex Iosevich)

Thank you for coming!