Vector valued polynomials, exponential polynomials and vector valued harmonic analysis

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Eötvös Loránd Univesity, Budapest

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$$\Delta_h f(x) = f(x+h) - f(x) \ (x \in G) \qquad (f\colon G \to \mathbb{C}).$$

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Definition

A continuous function $f : G \to \mathbb{C}$ is a generalized polynomial, if $(\exists n)$ s.t.

$$\Delta_{h_1} \ldots \Delta_{h_{n+1}} f = 0 \qquad (h_1, \ldots, h_{n+1} \in G).$$

The smallest such n is the degree of f.

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A continuous function $f\colon G \to \mathbb{C}$ *is a* *polynomial**, if* $f = P(\alpha_1, \ldots, \alpha_n)$*, where* $P \in \mathbb{C}[x_1, \ldots, x_n]$ *and* $\alpha_1, \ldots, \alpha_n$ *are continuous homomorphisms (additive functions) from* G *to* $\mathbb{C}$*.*

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A continuous function $f : G \to E$ is a generalized polynomial if and only if $u \circ f$ is a (complex valued) generalized polynomial for every $u \in E^*$.

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If G is a normed linear space and E is a Banach space, then $f : G \to E$ is a generalized polynomial if and only if $u \circ f$ is a (complex valued) generalized polynomial for every $u \in E^*$.

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Theorem

For every topological Abelian semigroup with unit G and for every Banach space E we have $\{$polynomials$\} \subset \{$w-polynomials$\} \subset \{$generalized polynomials$\} \subset \{$local polynomials$\}$

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$\{$polynomials$\}$ = $\{$w-polynomials$\}$ = $\{$generalized polynomials$\}$ = $\{$local polynomials$\}$.
In particular, this is true for $G = \mathbb{R}^n$ or $\mathbb{C}^n$.

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Theorem (S. Bochner and J. von Neumann, 1935)

Spectral synthesis holds in $C(G,E)$ for every compact Abelian group G and for every Banach space E.

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Let $f(g_n) = n! \cdot x_n$ for every n. Then V_f does not contain any function of the form $m \cdot e$, where m is an exponential and $e \in E$.

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Lemma

If $V \subset C(G, \mathbb{C}^k)$ is a variety, then the set of maps

$$\mathbb{Z}^k \times G \ni (t_1, \ldots, t_k, x) \mapsto t_1 f_1(x) + \ldots + t_k f_k(x) + g(x),$$

where $(f_1, \ldots, f_k) \in V$ and $g\colon G \to \mathbb{C}$ is a variety in $C(\mathbb{Z}^k \times G, \mathbb{C})$.

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If $r_0(G) < \infty$, then $r_0(\mathbb{Z}^k \times G) < \infty$.

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Is it true that if $r_0(G) < 2^\omega$ *and* k *is a positive integer, then spectral analysis holds in* $C(G, \mathbb{C}^k)$?

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Problem

Does there exist a locally compact and noncompact Abelian group G and a Banach space E of infinite dimension such that spectral synthesis holds in $C(G, E)$?

Thank you for your attention.