Finite Dimensional Varieties on Hypergroups

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Agenda

Spectral analysis and synthesis on vector modules

Let us consider a vector module.

A variety is a closed vector submodule.

Spectral analysis for a variety means that there are nonzero finite dimensional subvarieties in every nonzero variety.

Spectral synthesis means that there are sufficiently many nonzero finite dimensional varieties in every nonzero subvariety.

Our aim

What happens in the hypergroup-settings?

We investigate finite dimensional varieties invariant with respect to a compact subhypergroup K of a hypergroup X

Our assumption: (X, K) is a Gelfand pair, that means that the measure algebra of K-invariant measures is commutative.

Notation and terminology

Let $X = (X, *, \check{}, e)$ be a hypergroup. Let

$$\mathcal{C}(X) := \{f : X \to \mathbb{C} : f \text{ continuous } \}$$

be the locally convex topological vector with pointwise linear operations and the topology of compact convergence. The dual of $\mathcal{C}(X)$ can be identify with $\mathcal{M}_c(X)$, the space of all compactly supported complex measures on X, where the pairing is given by

$$\langle \mu, f \rangle = \int_X f d\mu$$

for each μ in $\mathcal{M}_c(X)$ and f in $\mathcal{C}(X)$.

Notation and terminology

Convolution on $\mathcal{M}_c(X)$ is given by

$$\langle \mu * \nu, f \rangle = \int_X \int_X f(x * y) d\mu(x) d\nu(y),$$

for any μ, ν in $\mathcal{M}_c(X)$ and f in $\mathcal{C}(X)$. Convolution of a measure μ from $\mathcal{M}_c(X)$ and a function f from $\mathcal{C}(X)$ is defined by

$$\mu * f(x) = \int_X f(x * \check{y}) d\mu(y), \quad , x \in X.$$

Notation and terminology

For any y in X and a continuous function $f: X \to \mathbb{C}$ we define the function $\tau_y f: X \to \mathbb{C}$ by the formula

$$\tau_y f(x) := f(x * y) := \int_X f(t) d(\delta_x * \delta_y)(t)$$

and call it the left translation of f by y.
A subset H of $\mathcal{C}(X)$ is called left-translation invariant, if for any f in H and any y in X the function $\tau_y f$ belongs to H.
A closed, left invariant subspace of $\mathcal{C}(X)$ is called a left variety.

Notation and terminology

Let K be a compact subhypergroup of the hypergroup X. The function f in $\mathcal{C}(X)$ is called K-invariant, if it satisfies

$$f(k * x * l) = f(x), \quad x \in X, k, l \in K.$$

The set of all K-invariant functions form a closed subspace of $\mathcal{C}(X)$ and it is denoted by $\mathcal{C}_K(X)$.

Notation and terminology

The projection $\mu^{\#}$ of the measure μ in $\mathcal{M}_c(X)$ is defined by

$$\langle \mu^{\#}, f\rangle = \int_X \int_K \int_K f(k * x * l) d\omega(k) d\omega(l) d\mu(x)$$

for each f in $\mathcal{C}(X)$. A measure μ in $\mathcal{M}_c(X)$ is called K-invariant if $\mu^{\#} = \mu$.

From now on if we say that "Let (X, K) be a Gelfand pair", then we mean that X is a hypergroup, $K \subseteq X$ is a compact subhypergroup, and (X, K) is a Gelfand pair, i.e. the algebra $\mathcal{M}_{c,K}(X)$ is commutative.

Notation and terminology

For every f in $\mathcal{C}_K(X)$ and for every y in X the K-invariant measure

$$D_{f;y} = \delta_{\check{y}}^{\#} - f(y)\delta_e$$

is called the modified K-spherical difference, where

$$\langle \delta_y^{\#}, f \rangle = \int_K \int_K f(k * y * l)d\omega(k)d\omega(l)$$

for every y in X. Moreover:

$$D_{f;y_1,\dots,y_{n+1}} := \prod_{j=1}^{n+1} D_{f;y_j}$$

for any natural number n and for each $y_1, \dots, y_{n+1}$ in X.

Notation and terminology

The non-zero K-invariant function $s\colon X \to \mathbb{C}$ is called a K-spherical function, if it satisfies

$$\int_K s(x * k * y)d\omega(k) = s(x)s(y) \tag{1}$$

for each x and y in X.

Notation and terminology

A subset H of $\mathcal{C}_K(X)$ is K-invariant, if for each f in H and y in X the function

$$\tau_y^{\#} f(x) = \int_K f(x * k * y) d\omega(k), \quad x \in X$$

is in H.
A closed K-invariant linear subspace of $\mathcal{C}_K(X)$ is K - variety.

Notation and terminology

A function f in $\mathcal{C}_K(X)$ is called a generalized K-monomial, if there exists a spherical function s and a natural number n such that

$$D_{s;y_1,\dots,y_{n+1}} * f(x) = \Big(\prod_{j=1}^{n+1} D_{s;y_j}\Big) * f(x) = 0$$

for each $x, y_1, \dots, y_{n+1}$ in X. If f is non-zero, then the spherical function s is unique and we call f a generalized spherical s - monomial and the smallest number n with the above property we call the degree of f.

Finte dimensional varieties

A K-variety in $\mathcal{C}_K(X)$ is called decomposable if it is the sum of two proper K-subvarieties. Otherwise the K-variety is called indecomposable.

Theorem 1

Let (X, K) be a Gelfand pair. Every finite dimensional K-variety can be decomposed into a finite sum of indecomposable K-varieties.

Finte dimensional varieties

Theorem 2

Let (X, K) be a Gelfand pair, d a positive integer. Suppose that $\Lambda : \mathcal{M}_{c,K}(X) \to \mathrm{M}(\mathbb{C}^{\mathrm{d}})$ is an algebra homomorphism. Then the matrix elements $x \mapsto \Lambda_{i,j}(\delta_x^{\#})$ are K-polynomials of degree at most d.

Corollary 1

Let (X, K) be a Gelfand pair. An indecomposable K-variety on X consists of s-monomials for some K-spherical function s.

Finte dimensional varieties

Corollary 2

Let (X, K) be a Gelfand pair. A K-variety on X is finite dimensional if and only if it is spanned by finitely many K-monomials.

Corollary 3

Let (X, K) be a Gelfand pair. Then the K-polynomials are exactly those continuous K-invariant functions on X, whose K-variety is finite dimensional.

Invariant differential operators

If $K \subseteq GL(\mathbb{R}^d)$ is a closed subgroup, then a differential operator D is called K -invariant, if

$$D(f \circ k) = Df \circ k$$

holds for each f in $\mathcal{C}^\infty(\mathbb{R}^d)$ and k in K.

Monomials on affine group

It turns out that all s_λ-monomials of the form $\partial^\alpha s$ with $|\alpha| \le N$ span the space of K-monomials of degree at most N:

Theorem 3

Every s_λ-monomial of degree at most N is a linear combination of the functions $\partial^\alpha s_\lambda$ with $|\alpha| \le N$.

Monomials on affine group

Corollary 4

Let $K \subseteq GL(\mathbb{R}^d)$ be a compact subgroup. Then the functions $\partial^\alpha s_\lambda$ with $|\alpha| \leq N$ form a basis of the linear space of s_λ-monomials of degree at most N.

Corollary 5

Let $K \subseteq GL(\mathbb{R}^d)$ be a compact subgroup. Then every K-monomial is analytic.

Theorem 4

Let $K \subseteq GL(\mathbb{R})$ be a compact subgroup and $V \neq \{0\}$ a finite dimensional K-variety. Then there exist positive integers p, different complex numbers λ_j, positive numbers m_j and K-invariant differential operators D_j $(j = 1, 2, \dots, p)$ such that V is the solution space of the differential equation

$$(D_1 - \lambda_1)^{m_1}(D_2 - \lambda_2)^{m_2} \cdots (D_p - \lambda_p)^{m_p} f = 0. \tag{2}$$

Special cases:

if $K = \{1\}$ we have $D_j = \frac{d}{dx}$,

if $K = \{-1, 1\}$ we have $D_j = \frac{d^2}{dx^2}$ for $j = 1, 2, \dots, p$.

In both cases the dimension of V is $\sum_{j=1}^{p} m_j$.

More examples

In the paper we also considered:

1. the polynomial hypergroup X.
2. hypergroup joins.
3. subhypergroup $K = SO(\mathbb{R}^d)$ of $GL(\mathbb{R}^d)$.

Summary

We have discussed

1. in general: finite dimensional varieties on hypergroups
2. special case: affine groups and connection to invariant differential operators
3. further examples

References

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Thank you very much for your kind attention