HARMONIC AND SPECTRAL ANALYSIS

International Zoom Conference

June 8–10 2020

Problems and Remarks

Problem no. 1
Characterization of Δ^m-invariant subspaces of measurable functions

JOSÉ MARÍA ALMIRA
University of Murcia

Let X denote either the space of continuous complex valued functions $\mathbf{C}(\mathbb{R})$ or the space of complex valued real distributions defined on the real line. In [1] the following theorems were demonstrated by using an algebraic technique [5] in combination with Anselone-Korevaar's theorem [2]:

Theorem 1. *Assume that V is a finite dimensional subspace of X which satisfies $\Delta_h^m(V) \subseteq V$ for all $h \in \mathbb{R}$. Then there exist vector spaces $\mathcal{P} \subset \mathbb{C}[t]$ and $\mathcal{E} \subset \mathbf{C}(\mathbb{R})$ such that*

$$V = \mathcal{P} \oplus \mathcal{E}$$

and $\mathcal{E}$ is invariant by translations. Consequently,

V is invariant by translations if and only if $\mathcal{P}$ is so.

Theorem 2. *Assume that V is a finite dimensional subspace of X. Then the following statements are equivalent:*

(i) $\Delta_h^m(V) \subseteq V$ *for all* $h \in \mathbb{R}$.

(ii) $\Delta_{h_1 h_2 \cdots h_m}(V) \subseteq V$ *for all* $(h_1, h_2, \cdots, h_m) \in \mathbb{R}^m$.

Note that Theorem 2 is, in certain sense, a generalization of the well known theorem by Djoković [3]. Moreover, to our knowledge, the generalization of these two theorems to higuer dimensional setting is still an open question (in particular, the proof given at [1] of Theorem 2 seems to be very difficult to translate to that setting).

On the other hand, the paper [4] contains an analogous result to Anselone-Korevaar's theorem for the case of measurable functions on σ-compact locally compact abelian groups. Thus, it would be interesting to demonstrate similar results to the Theorems 1 and 2 above, for the setting of Lebesgue measurable functions defined on the real line. We do not know if such results hold true or not (our conjecture is: no!)

In any case, if they hold true, the proofs should be different to the ones given in [1]. The reason for this last claim is the following one: In our proofs of Theorems 1, 2, we use the regularity assumptions (i.e., that elements f of V are either continuous functions or complex valued distributions) to guarantee that there exist a finite set of steps $h_1, \cdots, h_s$ such that, if $\Delta_{h_i}(V) \subseteq V$ for $i = 1, \cdots, s$, then $\Delta_h(V) \subset V$ for all h, and this could be untrue under the weaker assumption that the elements of V are measurable functions.

References

[1] J. M. Almira, Montel's Theorem and Subspaces of Distributions Which Are Δ^m-Invariant, *Numerical Functional Analysis and Optimization*, 35 (4) (2014) 389–403.

[2] P. M. Anselone, J. Korevaar, Translation invariant subspaces of finite dimension, *Proc. Amer. Math. Soc.* **15** (1964), 747–752.

[3] D. Z. Djoković, A representation theorem for $(X_1 - 1)(X_2 - 1)\cdots(X_n - 1)$ and its applications, *Ann. Polon. Math.* **22** (1969/1970) 189-198.

[4] M. Engert, Finite dimensional translation invariant subspaces, *Pacific J. Math.* **32** (2) (1970) 333–343.

[5] I. Gohberg, P. Lancaster, L. Rodman Invariant subspaces of matrices with applications, Classic in Applied Mathematics **51** SIAM, 2006.

Problem no. 2

STEFAN IVKOVIČ
The Mathematical Institute of the Serbian Academy of Sciences and Arts

Let $\mathcal{A}$ be a unital C^*-algebra and $H_{\mathcal{A}}$ denote the standard module over $\mathcal{A}$, that is $H_{\mathcal{A}} = l_2(\mathcal{A})$. We have the following proposition.

Proposition. *Let $\mathcal{A}$ be a unital C^*-algebra $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is unilateral shift and $S^*e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1))$ or if $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}$, (where in the case when $\mathcal{A} = L^\infty((0,1))$, we set $\inf|\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}[0,C]) > 0\} = \sup\{K > 0 \mid |\alpha| > K) \text{ a.e. on } (0,1)\}$). Moreover, $\sigma_p^{\mathcal{A}}(S) = \emptyset$ in both cases.*

Problem. *If $\mathcal{A} = B(H)$, can we give an explicit description of the generalized $\mathcal{A}$-valued spectrum of S as done in the proposition above?*

Consider again the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider bilateral shift operator V w.r.t. this basis i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^*e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition. *Let V be bilateral shift operator. Then the following holds*
1) If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \emptyset\}$
2) If $\mathcal{A} = L^\infty([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon)) > 0\ \forall \epsilon > 0\}$. In both cases $\sigma_p^{\mathcal{A}}(V) = \emptyset$.

Problem. *If $\mathcal{A} = B(H)$, can we give an explicit description of the generalized $\mathcal{A}$-valued spectrum of V in this case?*

Problem no. 3

Miklós Laczkovich
Eötvös Loránd University

Problem. *Does there exist a continuous function $f\colon \mathbb{R}^2 \to \mathbb{C}$ such that every linear combination of finitely many translates of f with complex coefficients has a root in the unit disc?*

Remark. It is easy to check that if f is such a function and V is the variety generated by f, then every element of V has a root in the closed unit disc. In particular, every element of V has a root, and thus V does not contain any exponential function. In this way we would obtain an example of a variety on $\mathbb{R}^2$ in which spectral analysis does not hold.

Problem no. 4

Gábor Somlai
Eötvös Loránd University and Alfréd Rényi Institute of Mathematics

The spectral-tile direction of Fuglede's conjecture for $\mathbb{Z}_p^3$ is open. This is just a combinatorial problem. We only have results for small primes ($p \leq 7$) and in those cases the answer is positive.

A spectral pair (A, B) in this case is a pair of subsets of $\mathbb{Z}_p^3$ such that for every $b_1 \neq b_2 \in B$ we have that A has $\frac{|A|}{p}$ elements on each translates of the plane orthogonal to $b_1 - b_2$.

The problem reduces to the following question.

Problem. *Does there exist a spectral pair (A, B) in $\mathbb{Z}_p^3$ with $p < |A| = |B| < p^2$?*

The positive answer is equivalent to the fact that the conjecture fails in this case.

Problem no. 5

Bettina Wilkens
University of Namibia

Let $n \in \mathbb{N}$, G a group, S a set, $f\colon G \to \mathcal{P}S$ be a subadditive map; assume that $|f(x)| \leq n$ for $x \in G$.

Let $T = \bigcup_{x \in G} f(x)$.

For $t \in T$, let $H_t = \{x \in G \mid t \notin f(x) \cup f(x^{-1})\}$. Each H_t is a proper subgroup of G.

The result that was mentioned in Professor Shulman's talk is the following

$$|T| \leq 2n \text{ and if the subgroups } H_t,\ t \in T, \text{ cover } G, \text{ then } |T| \leq 2n - 1.$$

Problem. *a) Describe the maps f satisfying $|T| = 2n$. Is there such a map that does not map all commutators to the empty set?*

From now on, we assume that $H = \bigcup_{t \in T} H_t$. B. Neumann's theorem says that G has a normal subgroup N of finite index in G with $f(x) = \emptyset$ whenever $x \in N$. This means that the we might as well assume that G is a finite group.

b) Given n, $|T| \leq 2n - s$ where s is the number of 1s in the binary representation of n. Which finite groups actually realise that bound? (Semi-answered).

b') If p is the smallest prime such that every element of G is a pth power in G, then $|T| \leq \frac{pn-1}{p-1}$, the bound is realised only when n is a p-power.
Anwer a "relative to p version" of b).

c) Let G be a finite group. Let $\mathcal{S}$ be an inclusion-independent set of proper subgroups of G. For $x \in G$, we let $f_{\mathcal{S}}(x) = \{H \in \mathcal{S} \mid x \notin H\}$, we let $\alpha(\mathcal{S}) = \max_{x \in G}(|\mathcal{S}| - |f_{\mathcal{S}}(x)|)$ and $\alpha(G)$ be the maximum value of $\alpha(\mathcal{S})$ with $\mathcal{S}$ ranging over the inclusion-independent sets of proper subgroups of G.
The general problem is "find $\alpha(G)$".
We have $\alpha(G) = 0$ if and only if G is cyclic.
Classifying or at least describing the finite groups with $\alpha(G) = 1$ is difficult, but possible and currently under way.

c') Is $\alpha(G) = 1$ *if and only if G is noncyclic and has an* uniserially embedded cyclic subgroup, *i.e. a subgroup* $\langle x \rangle$ *such that the subgroups of G containing x are totally ordered by inclusion?*