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The generalized spectra in C^*-algebras of operators over C^*-algebras

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Let $\mathcal{M}$ be a module over C^*-algebra $\mathcal{A}$. An action of an element $a \in \mathcal{A}$ on $\mathcal{M}$ is denoted by $x \cdot a$, where $x \in \mathcal{M}$.

Definition

[25, Definition 1.2.1.] A pre-Hilbert $\mathcal{A}$-module is a (right) $\mathcal{A}$-module M equipped with a sesquilinear form $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ with the following properties:
$(i)\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$;
$(ii)\langle x, x \rangle$ implies that $x = 0$;
$(iii)\langle y, x \rangle = \langle x, y \rangle^* a$ for any $x, y \in \mathcal{M}$;
$(iv)\langle x, ya \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$ and any $a \in \mathcal{A}$. The map $\langle \cdot, \cdot \rangle$ is called an $\mathcal{A}$-valued inner product.

Example

[25, Exammple 1.2.2] Let $J \subset \mathcal{A}$ be a right ideal. Then J can be equipped with the structure of a pre-Hilbert $\mathcal{A}$-module with the inner product of elements $x, y \in J$ defined by $\langle x, y\rangle := x^* y$.

Example

[25, Exammple 1.2.3] Let $\{J_i\}$ be a countable set of right ideals of a C^*-algebra $\mathcal{A}$ and let $\mathcal{M}$ be the linear space of all sequences $(x_i), x_{i \in J_i}$ satisfying the condition $\sum_i \parallel x_i \parallel^2 < \infty$. Then $\mathcal{M}$ becomes a right $\mathcal{A}$-module if the action of $\mathcal{A}$ is defined by $(x_i) \cdot a := (x_i a)$ for $(x_i) \in \mathcal{M}, a \in \mathcal{A}$, and becomes a pre-Hilbert $\mathcal{A}$-module if the inner product of elements $(x_i), (y_i) \in \mathcal{M}$ is defined by $\langle (x_i), (y_i)\rangle := \sum_i x_i^* y_i$.

Let $\mathcal{M}$ be a pre-Hilbert $\mathcal{A}$-module, $x \in \mathcal{M}$. Put $\| x \|_{\mathcal{M}} := \| \langle x, x \rangle \|^{\frac{1}{2}}$. We usually skip the subscript $\mathcal{M}$ when it does not lead to confusion of norms.

Proposition

[25, Proposition 1.2.4] The function $\| \cdot \|_{\mathcal{M}}$ is a norm on $\mathcal{M}$ and satisfies the following properties:

(i) $\| x \cdot a \|_{\mathcal{M}} \leq \| x \|_{\mathcal{M}} \cdot \| a \|$ for any $x \in \mathcal{M}, a \in \mathcal{A}$;

(ii) $\langle x, y \rangle \langle y, x \rangle \leq \| y \|_{\mathcal{M}}^2 \langle x, x \rangle$ for any $x, y \in \mathcal{M}$;

(iii) $\| \langle x, y \rangle \| \leq \| x \|_{\mathcal{M}} \| y \|_{\mathcal{M}}$ for any $x, y \in \mathcal{M}$.

Definition

[25, Definition 1.3.2] A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a Hilbert C^*-module is it is complete with respect to the norm $\| \cdot \|_{\mathcal{M}}$.

Example

[25, Example 1.3.3] If $J \subset \mathcal{A}$ is a closed right ideal, then the pre-Hilbert module J is complete with respect to the norm $\| \cdot \|_J = \| \cdot \|$. In particulaar, the C^*-algebra $\mathcal{A}$ itself is a free Hilbert $\mathcal{A}$-module with one generator.

Example

[25, Example 1.3.5] If $\{\mathcal{M}_i\}, i \in \mathbb{N}$, is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus \mathcal{M}_i$. On the $\mathcal{A}$-module $\oplus \mathcal{M}_i$ of all sequences $x = (x_i) : x_i \in \mathcal{M}_i$, such that the series $\sum_i \langle x_i, y_i \rangle$ is norm-convergent in the C^*-algebra $\mathcal{A}$, we define the inner product by $\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle$ for $x, y \in \oplus \mathcal{M}_i$. If each $\mathcal{M}_i = \mathcal{A}$, then will denote $\oplus \mathcal{M}_i$ by $H_{\mathcal{A}}$. This module is called the standard module over $\mathcal{A}$. So $H_{\mathcal{A}} = l_2(\mathcal{A})$. If $\mathcal{A}$ is unital, then $H_{\mathcal{A}} = l_2(\mathcal{A})$ has natural orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$.

Let $\mathcal{N} \subset \mathcal{M}$ be a closed submodule of a Hilbert C^*-module $\mathcal{M}$. We define the orthogonal complement $\mathcal{N}^\perp$ by the formula.

$$\mathcal{N}^\perp = \{y \in \mathcal{M} : \langle x, y \rangle = 0 \text{ for all } x \in \mathcal{N}\}.$$

Then $\mathcal{N}^\perp$ is a closed submodule of the Hilbert C^*-module $\mathcal{M}$ too. However, the equality $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^\perp$ does not always hold, as the following example shows.

Example

[25, Example 1.3.7] Let $\mathcal{A} = C[0,1]$ be the C^*-algebra of all continuous function on the segment $[0,1]$. Consider, in the Hilbert $\mathcal{A}$-module $\mathcal{M} = \mathcal{A}$, the submodule $\mathcal{N} = C_0(0,1)$ of functions that vanish at the end points of the segment. Then, obviously, $\mathcal{N}^\perp = 0$.

If M and N are two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$, then a map $T : M \to N$ is called an $\mathcal{A}$-linear operator if $T(x \cdot \alpha) = T(x) \cdot \alpha$ for all $x \in M$ and $\alpha \in \mathcal{A}$. In particular this means that T is linear because $T(\lambda x) = T(x \cdot \lambda 1) = Tx \cdot \lambda 1 = \lambda Tx$ for all $\lambda \in \mathbb{C}$. The set of all bounded, $\mathcal{A}$-linear operators from M into N will be denoted by $B(M, N)$. In addition, an operator $T \in B(M, N)$ is said to be adjointable if there exists an $\mathcal{A}$-linear operator $T^* : N \to M$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in M, y \in N$. It turns out that in this case T^* is also bounded. The set of all adjointable, bounded, $\mathcal{A}$-linear operators from M into N will be denoted by $B^a(M, N)$. $B^a(M, N)$ is a C^*-algebra.

Presentation

Question: If $\mathcal{A}$ is a C^*-algebra, then for $\alpha \in \mathcal{A}$ could we define in a suitable way the operator αI on $H_{\mathcal{A}}$ and the generalized spectra in $\mathcal{A}$ of operators in $B^a(H_{\mathcal{A}})$ by setting for every $F \in B^a(H_{\mathcal{A}})$
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in} B^a(H_{\mathcal{A}})\}$?
Answer: For $\alpha \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by $\alpha I(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots)$. It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\| \alpha I \| = \| \alpha \|$.
Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.
We introduce then the following notion:
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in} B^a(H_{\mathcal{A}})\}$;
$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \ker(F - \alpha I) \neq \{0\}\}$;
$\sigma_{rl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is bounded below, but not surjective on } H_{\mathcal{A}})\}$;
$\sigma_{cl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid Im(F - \alpha I) \text{ is } \underline{\text{not}} \text{ closed } \}$. (where $F \in B^a(H_{\mathcal{A}})$)).

Remark

As we have seen, it turns out that not all closed submodules of $H_{\mathcal{A}}$ are orthogonally complementable in $H_{\mathcal{A}}$ as we have in the situation of Hilbert spaces. So it may happen that $\overline{Im(F - \alpha I)} \oplus Im(F - \alpha I)^{\perp} \subsetneqq H_{\mathcal{A}}$. However, if $Im(F - \alpha I)$ is closed, then $Im(F^* - \alpha^* I)$ is closed and we also have $H_{\mathcal{A}} = Im(F - \alpha I) \oplus \ker(F^* - \alpha^* I) = \ker(F - \alpha I) \oplus Im(F^* - \alpha^* I)$ (see Theorem 2.3.3. in the book 'Hilbert C^*-modules' by V.M.Manuilov and E.V.Troitsky) whenever $F \in B^a(H_{\mathcal{A}})$. Therefore, it is more convinient in this setting to work with $\sigma_{rl}^{\mathcal{A}}(F)$ and $\sigma_{cl}^{\mathcal{A}}(F)$ for $F \in B^a(H_{\mathcal{A}})$ instead of the residual and the continuous spectrum.

Remark

We have obviously $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$ and $\sigma^{\mathcal{A}}(F^*) = (\sigma^{\mathcal{A}}(F))^*$.

Challenges:

1) $\mathcal{A}$ may be non commutative.
2) Not all non-zero elements in $\mathcal{A}$ are invertible. Moreover, even if $\alpha \in \mathcal{A} \cap G(\mathcal{A})$, we do not have in general that $\| \alpha^{-1} \| = \frac{1}{\| \alpha \|}$. Therefore $\sigma^{\mathcal{A}}(F)$ may be unbounded. (However $\sigma^{\mathcal{A}}(F)$ is always closed in $\mathcal{A}$).

Proposition

Let $\mathcal{A}$ be a unital C^*-algebra, $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is unilateral shift and $S^* e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1))$ or if $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$, (where in the case when $\mathcal{A} = L^\infty((0,1))$, we set $\inf |\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}[0,C]) > 0\} = \sup\{K > 0 \mid |\alpha| > K)$ a.e. on $(0,1)\}$). Moreover, $\sigma_p^{\mathcal{A}}(S) = \varnothing$ in both cases.

Example

Let $\mathcal{A} = L^\infty((0,1))$. Set $\tilde{S}(f_1, f_2, \cdots) = (f_1\chi_{(0,\frac{1}{2})}, f_2\chi_{(0,\frac{1}{2})} + f_1\chi_{(\frac{1}{2},1)}, f_3\chi_{(0,\frac{1}{2})} + f_2\chi_{(\frac{1}{2},1)}, \cdots)$. Then $\tilde{S}$ has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$ w.r.t. the decomposition $(H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)})$. It follows that
$\sigma^{\mathcal{A}}(\tilde{S}) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \cdot \chi_{(\frac{1}{2},1)} \leq 1\}$
$\cup\{\alpha \in \mathcal{A} \mid \mu(\{t \mid t \in (0, \frac{1}{2})$ and $\alpha(t) = 1\}) > 0\}$.

Proposition
Let $\alpha \in \mathcal{A}$. We have
1. If $\alpha I - F$ is bounded below, and $F \in B^a(H_{\mathcal{A}})$ then $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$.
2. If $F, D \in B^a(H_{\mathcal{A}})$ and $D = U^* F U$ for some unitary operator U, then $\sigma^{\mathcal{A}}(F) = \sigma^{\mathcal{A}}(D), \sigma_p^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(D), \sigma_{cl}^{\mathcal{A}}(F) = \sigma_{cl}^{\mathcal{A}}(D)$ and $\sigma_{rl}^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(D)$.

Proposition
Let $U \in B^a(H_{\mathcal{A}})$ be unitary. Then $\sigma^{\mathcal{A}}(U) \subseteq \{\alpha \in \mathcal{A} \mid \| \alpha \| \geq 1\}$ and $\sigma^{\mathcal{A}}(U) \cap G(\mathcal{A}) \subseteq \{\alpha \in G(\mathcal{A}) \mid \| \alpha^{-1} \|, \| \alpha \| \geq 1\}$.

Proof.
We have $\alpha I - V = (\alpha V^* - I)V$ and $\| V^* \| = \| V \| = 1$. □

Consider again the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider bilateral shift operator V w.r.t. this basis i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^*e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition

Let V be bilateral shift operator. Then the following holds

1) If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \varnothing\}$

2) If $\mathcal{A} = L^{\infty}([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon)) > 0 \ \forall \epsilon > 0\}$. In both cases $\sigma_p^{\mathcal{A}}(V) = \varnothing$.

Example

Let $\{\alpha_1, \alpha_2, \cdots\}$ be a sequence in a unital C^*-algebra $\mathcal{A}$ s.t. each α_k is a unitary element of $\mathcal{A}$. Then the operator V defined by $V(x_1, x_2, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots)$ is a unitary operator on $H_{\mathcal{A}}$. Then $\sigma^{\mathcal{A}}(V) = \{\beta \in \mathcal{A} \mid \beta - \alpha_k$ is not right invertible in $\mathcal{A}$ or that $(\beta - \alpha_k)\gamma_k = 0$ for some $\gamma_k \in \mathcal{A}\}$. If $\mathcal{A} = C([0,1])$ or if $\mathcal{A} = L^\infty((0,1))$ and J_1, J_2 are two closed subintervals of $(0,1)$ such that $J_1 \cap J_2 = \varnothing$, then we may easily find a function $\beta \in \mathcal{A}$ such that $\beta = \alpha_1$ on J_1 and $|\beta(t)| > 1$ for all $t \in J_2$. Hence $\| \beta \| > 1$, but we also have $\beta \in \sigma^{\mathcal{A}}(V)$. Similarly, if $\mathcal{A} = B(H)$ where H is a Hilbert space, then we may easily find two closed suspaces H_1 and H_2 such that $H_1 \perp H_2$ and $T \in B(H)$ is such that $T_{|_{H_1}} = \alpha_{1|_{H_1}}$ and $\| T_{|_{H_2}} \| > 1$. Hence again $T \in \sigma^{\mathcal{A}}(V)$ and $\| T \| > 1$. So, if V is a unitary operator on $H_{\mathcal{A}}$, we do <u>not</u> have in general that $\sigma^{\mathcal{A}}(V) \subseteq \{\alpha \in \mathcal{A} \mid \| \alpha \| = 1\}$.

Lemma

If F is a self-adjoint operator on $H_{\mathcal{A}}$, then $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$, that is $\alpha \in \sigma_p^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F)$ in the case when $\mathcal{A}$ is a commutative C^*-algebra.

Proof.

Since $F - \alpha I$ and $F - \alpha^* I = F^* - \alpha^* I$ mutually commute, we can deduce that $\| (F - \alpha I)x \| = \| (F - \alpha^* I)x \|$ for all $x \in H_{\mathcal{A}}$. □

Example

Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^{\infty}((0,1))$. If G is the operator on $H_{\mathcal{A}}$ given by $G(f_1, f_2, \cdots) = (g_1 f_1, g_2 f_2, \cdots)$, where $\{g_1, g_2, \cdots\}$ is a bounded sequence of real valued functions in $\mathcal{A}$, then G is a self-adjoint operator. Suppose that there are two mutually disjoint, closed subintervals J_1 and J_2 of $(0,1)$ such that ${g_1}_{|_{J_1}} \neq 0$ and ${g_1}_{|_{J_2}} = 0$. Set $\tilde{g} = i g_1$. Then, if we choose a function f in $\mathcal{A}$ such that $\operatorname{supp} f \subseteq J_2$, we get that $(\tilde{g} I - G)(f, 0, 0, \cdots) = 0$. However $\tilde{g} \neq \overline{\tilde{g}}$, so we do <u>not</u> have that $\sigma_p^{\mathcal{A}}(G)$ is included in the set of self-adjoint elements of $\mathcal{A}$.

Example

Let $\mathcal{A} = B(H)$ where H is a Hilbert space and let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for H. If P denotes the orthogonal projection onto $Span\{e_1\}$, then the operator $P \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$. Now, if S is the unilateral shift operator on H w.r.t. to the orthonormal basis $\{e_j\}$, then $S - P$ is injective whereas $S^* - P$ is not injective because $(S^* - P)(e_1 + e_2) = 0$. It follows that $(S - P) \cdot I$ is an injective operator whereas $(S^* - P) \cdot I = ((S - P) \cdot I)^*$ is **not** an injective operator. Hence, if $\mathcal{A} = B(H)$ where H is a Hilbert space, we do not have in general that $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$ when $F = F^*$.

Lemma

Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, then $\overline{R(F - \alpha I)}^{\perp} = \{0\}$. Hence, if $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$ and in addition $F - \alpha I$ is bounded below, then $\alpha \in \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.

Corollary

Let $\mathcal{A}$ be a unital commutative C^*-algebra and F be a self-adjoint operator on $H_{\mathcal{A}}$. If $\alpha \in \mathcal{A}$ and $\alpha - \alpha^* \in G(\mathcal{A})$, then $F - \alpha I$ is invertible. In this case $\| (F - \alpha I)^{-1} \| \leq 2 \| (\alpha - \alpha^*)^{-1} \|$.

Remark

Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^{\infty}((0,1))$. As we have seen in Example the operator $\tilde{g}I - G$ is not invertible, whereas $\tilde{g} - \overline{\tilde{g}} = 2ig_1 \neq 0$. Therefore, the requirement that $\alpha - \alpha^*$ is invertible is indeed necessary in Corollary.

Example

Let $\mathcal{A} = M_2(\mathbb{C})$ and $T_1, T_2 \in \mathcal{A}$ be given as $T_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & i \\ i & i \end{bmatrix}$. Then T_1 is self-adjoint and $T_2 - T_2^* = 2i \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, so $T_2 - T_2^*$ is invertible. Now $T_1 - T_2 = \begin{bmatrix} 1 & 1-i \\ 1-i & -i \end{bmatrix}$, so $det(T_1 - T_2) = 0$ which gives $T_1 - T_2$ is not invertible. Hence the operator $F := T_1 \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$, but $F - T_2 \cdot I = (T_1 - T_2) \cdot I$ is <u>not</u> invertible. This shows that the assumption that $\mathcal{A}$ is commutative in Corollary is indeed necessary.

For a self-adjoint operator F on $H_{\mathcal{A}}$, set
$M(F) = \sup\{\| \langle Fx, x\rangle \| \| \| x \|= 1\}$ and
$m(F) = \inf\{\| \langle Fx, x\rangle \| \| \| x \|= 1\}$. We have the following corollary.

Corollary

If $\mathcal{A} = C([0,1])$ *and* F *is a self-adjoint operator on* $H_{\mathcal{A}}$*, then* $\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid |f|(0,1) \cap [m, M] \neq \varnothing\}$. *If* $\mathcal{A} = L^{\infty}((0,1))$ *and* F *is a self-adjoint operator on* $H_{\mathcal{A}}$ *then*
$\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid \mu(|f|^{-1}[m-\epsilon, M+\epsilon]) > 0 \text{ for some } \epsilon = \epsilon(f) > 0\}$.

Lemma

Let $\mathcal{A}$ be a commutative unital C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$, that is $FF^* = F^*F$. If $\alpha_1, \alpha_2 \in \sigma_p^{\mathcal{A}}(F)$ and $\alpha_1 - \alpha_2$ is invertible in $\mathcal{A}$, then $\ker(F - \alpha_1 I) \perp \ker(F - \alpha_2 I)$.

Example

Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1))$, consider the self-adjoint operator G from the previous example. For any function f in $\mathcal{A}$ with the support contained in J_2, we have $(f, 0, 0, \cdots) \in \ker G \cap \ker(\tilde{g}I - G)$. However $\tilde{g} = ig_1 \neq 0$ and $f \neq 0$, but $\tilde{g}$ is <u>not</u> invertible in $\mathcal{A}$.

Example

Let $\mathcal{A} = B(H)$ and $T \in \mathcal{A}$ be a normal and invertible operator. If H_1 and H_2 are two closed subspaces of H such that $H = H_1 \tilde{\oplus} H_2$ and $H_1 \neq H_2^{\perp}$ (that is H_1 and H_2 are not mutually orthogonal), then $T\Pi$ and $T(1-\Pi)$ are elements of $\sigma_p^{\mathcal{A}}(T \cdot I)$, where Π stands for the skew projection onto H_1 along H_2. Moreover, the operator $T \cdot I$ is normal operator on $H_{\mathcal{A}}$ and $T\Pi - T(1-\Pi)$ is invertible in $\mathcal{A}$. However, if P_1 and P_2 denote the orthogonal projections onto H_1 and H_2, respectively, then $e_j \cdot P_1 \in \ker(T\Pi \cdot I - T \cdot I), e_j \cdot P_2 \in \ker(T(I-\Pi) \cdot I - T \cdot I)$ for all j and $P_1 P_2 \neq 0$. So the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma.

Lemma

Let $\mathcal{A}$ be a commutative C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$. Then $\sigma_{rl}^{\mathcal{A}}(F) = \varnothing$, hence $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Example

Let $\mathcal{A} = B(H)$ and S, P be as in Example. Then $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ being self-adjoint and $(S - P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. Indeed, $\| (S - P)x \| \geq \| x \|$ for all $x \in H$, hence $m(S - P) \geq 1$ so $T^*(S - P)^*(S - P)T \geq m(S - P)^2 T^* T$ for all $T \in B(H)$ which gives that $(S - P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. However, $Im((S - P) \cdot I)^{\perp} = \ker((S^* - P) \cdot I)$. and $\ker((S^* - P) \cdot I) \neq \{0\}$ as we have seen in Example. Hence $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ and $S \in \sigma_{rl}^{\mathcal{A}}(P \cdot I)$ which shows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma.

Next, for $F \in B^a(H_{\mathcal{A}})$, set $\sigma_a^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I$ is not bounded below $\}$.

Proposition

For $F \in B^a(H_{\mathcal{A}})$, we have that $\sigma_a^{\mathcal{A}}(F)$ is a closed subset of $\mathcal{A}$ in the norm topology and $\sigma^{\mathcal{A}}(F) = \sigma_a^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F)$.

Proposition

If $F \in B^a(H_{\mathcal{A}})$, then $\partial\sigma^{\mathcal{A}}(F) \subseteq \sigma_a^{\mathcal{A}}(F)$. Moreover, if M is a closed submodule of $H_{\mathcal{A}}$ and invariant with respect to F, and $F_0 = F_{|_M}$, then we have $\partial\sigma^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F)$, $\sigma^{\mathcal{A}}(F_0) \cap \sigma^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(F_0)$.

Example

We may also consider the operators on $H_{\mathcal{A}}$ when $\mathcal{A}$ is a unital C^*-algebra defined as $W'(e_k) = e_{2k}$ and $W''(e_k) = e_{2k-1}$ $\forall k \in \mathbb{N}$. Also for these operators we have $\sigma^{\mathcal{A}}(W') = \sigma^{\mathcal{A}}(W'') = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$ in the case when $\mathcal{A} = C([0,1])$ or when $\mathcal{A} = L^{\infty}((0,1))$. Suppose now that $\mathcal{A} = L^{\infty}((0,1))$ and consider the operator F on $H_{\mathcal{A}}$ given by $F(f_1, f_2, f_3, \cdots) = (\chi_{(0,\frac{1}{2})})f_1, \chi_{(\frac{1}{2},1)}f_1, \chi_{(0,\frac{1}{2})})f_2, \chi_{(\frac{1}{2},1)}f_2, \cdots)$. It follows that F has the matrix $\begin{bmatrix} W'' & 0 \\ 0 & W' \end{bmatrix}$ w.r.t the decomposition $(H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)})$. Therefore $\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$.
Next $\sigma_p^{\mathcal{A}}(W') = \varnothing$,
$\sigma_p^{\mathcal{A}}(W'') = \{\alpha \in \mathcal{A} \mid \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}$ in the case when $\mathcal{A} = C([0,1])$ and
$\sigma_p^{\mathcal{A}}(W'') = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0,1) \mid \alpha(t) = 1\}) > 0\}$ in the case $\mathcal{A} = L^{\infty}((0,1))$. Hence
$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0,\frac{1}{2}) \mid \alpha(t) = 1\}) > 0\}$.

Consider now the operators

$$Z(e_j) = \begin{cases} e_k & \text{when } j = 2k \\ 0 & \text{else} \end{cases}, \ k \in \mathbb{N}$$

$$Z'(e_j) = \begin{cases} e_k & \text{when } j = 2k-1 \\ 0 & \text{else} \end{cases}, \ k \in \mathbb{N}$$

Then $\sigma^{\mathcal{A}}(Z) = \sigma^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}$. This follows since $Z = W'^{*} and\ Z' = W''^{*}$. Moreover, we have $\sigma_p^{\mathcal{A}}(Z) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1\}$ in both cases. In the case $\mathcal{A} = L^{\infty}((0,1))$ we have $\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \mu(\{t \in (0,1) \mid \alpha(t) = 1)\} > 0\}$ and in the case when $\mathcal{A} = C([0,1])$ we have $\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}$. Let operator D on $H_{\mathcal{A}}$ be given by

$D(g_1, g_2, g_3, \cdots) = (g_1\chi_{(0,\frac{1}{2})} + g_2\chi_{(\frac{1}{2},1)}, g_3\chi_{(0,\frac{1}{2})}) + g_4\chi_{(\frac{1}{2},1)}, \cdots)$ when $\mathcal{A} = L^{\infty}((0,1))$. Then $D = F^*$ and D has the matrix $\begin{bmatrix} Z' & 0 \\ 0 & Z \end{bmatrix}$ w.r.t. the decomposition $H_{\mathcal{A}}\chi_{(0,\frac{1}{2})} \oplus H_{\mathcal{A}}\chi_{(\frac{1}{2},1)}$ It follows that

$\sigma^{\mathcal{A}}(D) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}$ and

$\sigma_p^{\mathcal{A}}(D) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \mu(\{t \in (0,\frac{1}{2}) \mid \alpha(t) = 1)\} > 0\}$.

Thank you for attention!

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